ON WEIGHTED DENSITIES

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Abstract. Results on the continuity of densities given by the weight function n^α, $\alpha \in [-1, \infty[$, with respect to the parameter α are presented.

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1. Introduction

Let $f: \mathbb{N} \to [0, \infty[$ be a nonzero function, let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. We denote

$$A_f(n) = \sum_{a \in A, a \leq n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n \to \infty} \frac{A_f(n)}{\mathbb{N}_f(n)} \quad \text{and} \quad \overline{d}_f(A) = \limsup_{n \to \infty} \frac{A_f(n)}{\mathbb{N}_f(n)},$$

i.e. the lower and the upper f-densities of the set A, respectively. Put moreover

$$D_f(A) = (\overline{d}_f(A), \underline{d}_f(A)) \in \{(x, y);\ x \in [0,1],\ y \in [0, x]\}.$$

We call $D_f(A)$ the *f-density point* of the set A. Two important cases of densities are those of asymptotic densities (denoted by $\underline{d}, \overline{d}$) with $f(n) = 1$, $n \in \mathbb{N}$, and logarithmic densities (denoted by $\underline{\delta}, \overline{\delta}$) with $f(n) = \frac{1}{n}$, $n \in \mathbb{N}$. There is a well known relation among these four values (see, for instance, [H–Ro], p. 241-242)

(I) $$0 \leq \underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A) \leq 1$$

which holds for every $A \subset \mathbb{N}$. Also there are known examples of sets for which values of the asymptotic densities differ from the corresponding ones of the logarithmic densities. Even there exist sets with arbitrary prescribed values of all

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four densities respecting the relation (I) (see [M]). In this paper we will deal with the class of densities determined by weight functions $f_\alpha(n) = n^\alpha$, $\alpha \in [-1,\infty[$. Notice that the asymptotic densities correspond to $\alpha = 0$ and the logarithmic densities correspond to $\alpha = -1$.
In the sequel we shall write A_α in place of A_{f_α} and $\underline{d}_\alpha$, $\overline{d}_\alpha$, D_α in place of $\underline{d}_{f_\alpha}$, $\overline{d}_{f_\alpha}$ and D_{f_α}, respectively, and moreover we shall use the term α-density point instead of f_α-density point.
In [R] it is proved that both the upper and lower α-densities vary monotonously with respect to the parameter α. This provides an extension of inequalities (I): Let $-1 \le \alpha < \beta < \infty$. Then the inequalities

$$\text{(R)} \qquad \underline{d}_\beta(A) \le \underline{d}_\alpha(A) \quad \text{and} \quad \overline{d}_\alpha(A) \le \overline{d}_\beta(A)$$

hold for every $A \subset \mathbb{N}$.
A natural question arises whether the coordinates of the α-density point of the set A depend on the parameter α continuously. The aim of the present paper is to discuss this question. As there are no well known examples of sets with different α-density points for $\alpha \in [-1,\infty[$, we will start with the following example. It shows that there are sets $A \subset \mathbb{N}$ for which both functions $\alpha \mapsto \underline{d}_\alpha(A)$ and $\alpha \mapsto \overline{d}_\alpha(A)$ are injective on $[-1,\infty[$.

Example 1. *Let $a > 1$ be a real number. Denote by $A = \bigcup_{k=0}^{\infty}][a^{2k}],[a^{2k+1}]] \bigcap \mathbb{N}$, where $[r]$ means the integer part of the real number r, i.e. the largest integer less than or equal to r. Then for every $\alpha \in [-1,\infty[$*

$$\underline{d}_\alpha(A) = \frac{1}{a^{\alpha+1}+1} \quad \textit{and} \quad \overline{d}_\alpha(A) = \frac{a^{\alpha+1}}{a^{\alpha+1}+1}.$$

First, let $\alpha > -1$. Then both densities can be calculated using the technique in [M–T], integrating the function x^α in corresponding intervals and cancelling the constant multipliers $\frac{1}{\alpha+1}$

$$\overline{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum\limits_{k=0}^{n} \sum\limits_{i=[a^{2k}]+1}^{[a^{2k+1}]} i^\alpha}{\sum\limits_{j=1}^{a^{2n+1}} j^\alpha} = \limsup_{n\to\infty} \frac{\sum\limits_{k=0}^{n} (a^{2k+1})^{\alpha+1} - (a^{2k})^{\alpha+1}}{(a^{2n+1})^{\alpha+1}} =$$

$$= \lim_{n\to\infty} (a^{\alpha+1}-1) \frac{\sum\limits_{k=0}^{n} (a^{2\alpha+2})^k}{(a^{2n+1})^{\alpha+1}} = (a^{\alpha+1}-1) \lim_{n\to\infty} \frac{\frac{(a^{2\alpha+2})^{n+1}-1}{a^{2\alpha+2}-1}}{(a^{2n+1})^{\alpha+1}} =$$

$$= \frac{1}{a^{\alpha+1}+1} \lim_{n\to\infty} \frac{a^{2\alpha n+2n+2\alpha+2}}{a^{2\alpha n+2n+\alpha+1}} = \frac{a^{\alpha+1}}{a^{\alpha+1}+1}$$

and, similarly, or using the fact that in this case $\underline{d}_\alpha(A) = \frac{\overline{d}_\alpha(A)}{a^{\alpha+1}}$, we get

$$\underline{d}_\alpha(A) = \frac{1}{a^{\alpha+1}+1}.$$

Calculation of $\underline{d}_{-1}(A)$ and $\overline{d}_{-1}(A)$ can be performed using the same technique to get

$$\underline{d}_{-1}(A) = \frac{1}{2} = \overline{d}_{-1}(A).$$

Notice that the same result can be obtained using Theorem 2 below on continuity at $\alpha = -1$, as the set A fulfils its assumptions and

$$\lim_{\alpha\to -1^+} \frac{a^{\alpha+1}}{a^{\alpha+1}+1} = \lim_{\alpha\to -1^+} \frac{1}{a^{\alpha+1}+1} = \frac{1}{2}.$$

2. Continuity on $]-1,\infty[$

We are now going to answer to the question about the continuity of dependence of α- density points on the parameter α. First we will consider the case $\alpha \in]-1,\infty[$.

Theorem 1. *Let $\alpha \in]-1,\infty[$ and $\delta > 0$. Then for every set $A \subset \mathbb{N}$*

$$|\underline{d}_\alpha(A) - \underline{d}_{\alpha+\delta}(A)| \le \frac{2\delta}{\alpha+1} \quad \textit{and} \quad |\overline{d}_\alpha(A) - \overline{d}_{\alpha+\delta}(A)| \le \frac{2\delta}{\alpha+1}.$$

Remark 1. *Since for all $\alpha > -1$ and all δ such that $0 < \delta < \alpha+1$ we have $\alpha - \delta > -1$, the statement of the theorem can be applied for the pair $a - \delta > -1$ and $\alpha = (\alpha-\delta)+\delta$ to get*

$$|\underline{d}_\alpha(A) - \underline{d}_{\alpha-\delta}(A)| \le \frac{2\delta}{\alpha-\delta+1} \quad \textit{and} \quad |\overline{d}_\alpha(A) - \overline{d}_{\alpha-\delta}(A)| \le \frac{2\delta}{\alpha-\delta+1}$$

for all $A \subset \mathbb{N}$.

Thus we have direct consequences of the above theorem.

Corollary 1. *Given a set $A \subset \mathbb{N}$, the function $\alpha \mapsto D_\alpha(A)$ is Lipschitzian on each closed half-line $[a_0,\infty[$, with $a_0 > -1$ fixed.*

Corollary 2. *Given a set $A \subset \mathbb{N}$, the function $\alpha \mapsto D_\alpha(A)$ is continuous on $]-1,\infty[$.*

3. The continuity at -1

Let A be a fixed subset of $\mathbb{N}$. In this section we shall study the continuity of the α-density points as $\alpha \to -1^+$. We assume that the set $A \subseteq \mathbb{N}$ is neither finite nor cofinite, so that it can be written in the form

$$A = \mathbb{N} \cap (\bigcup_{n=1}^{\infty}]a_n, b_n])$$

for two suitable sequences of integers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $a_n < b_n < a_{n+1}$ for every n. We recall that

$$\mathbb{N}_\alpha(n) = \sum_{k=1}^{n} k^\alpha, \qquad n \in \mathbb{N}.$$

By an application of Theorem 8.2 of [F–GA], we are able to calculate the upper and lower α-densities of A as follows:

Theorem A.. *The following relations hold*

$$\underline{d}_\alpha(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1}(\mathbb{N}_\alpha(b_k) - \mathbb{N}_\alpha(a_k))}{\mathbb{N}_\alpha(a_n)}$$

$$\overline{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(\mathbb{N}_\alpha(b_k) - \mathbb{N}_\alpha(a_k))}{\mathbb{N}_\alpha(b_n)}. \tag{1}$$

By the equivalence relations, as $n \to \infty$

$$n^\alpha \sim \begin{cases} (1+\alpha)((n+1)^\alpha - n^\alpha) & \text{for } \alpha > -1 \\ \log(n+1) - \log n & \text{for } \alpha = -1 \end{cases}$$

it is easily seen that, for $\alpha > -1$, $\overline{d}_\alpha(A)$ (resp. $\underline{d}_\alpha(A)$) can be also calculated as

$$\overline{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(b_k^{1+\alpha} - a_k^{1+\alpha})}{b_n^{1+\alpha}}, \tag{2}$$

(resp.

$$\underline{d}_\alpha(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1}(b_k^{1+\alpha} - a_k^{1+\alpha})}{a_n^{1+\alpha}}), \tag{3}$$

while, for $\alpha = -1$ we have

$$\overline{d}_{-1}(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(\log b_k - \log a_k)}{\log b_n}, \tag{4}$$

(resp.

$$\underline{d}_{-1}(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1}(\log b_k - \log a_k)}{\log a_n}). \tag{5}$$

In the sequel we set, for each n,

$$C_n = \log b_n - \log a_n; \quad B_n = \log b_n - \log b_{n-1}; \quad A_n = \log a_n - \log a_{n-1};$$

we shall suppose that the sequence $(B_n)_{n\geq 1}$ is bounded (assumption (H)). This easily implies that $(A_n)_{n\geq 1}$ and $(C_n)_{n\geq 1}$ are bounded as well.

We have the following result

Theorem 2. *In addition to assumption (H), suppose that*

$$L \doteq \liminf_{n\to\infty} C_n > 0 \tag{6}$$

Then we have

$$\lim_{\alpha\to -1^+} \overline{d}_\alpha(A) = \overline{d}_{-1}(A). \tag{7}$$

$$\lim_{\alpha\to -1^+} \underline{d}_\alpha(A) = \underline{d}_{-1}(A). \tag{8}$$

The following example shows that assumption (H) cannot be dropped.

Example 2. *For the set* $A = \mathbb{N} \cap (\bigcup_{n=1}^{\infty}]a_n, b_n])$, *with*

$$a_n = n((n-1)!)^2, \qquad b_n = (n!)^2,$$

assumption (H) is not verified.

In fact we have

$$C_n = \log(n!)^2 - \log n((n-1)!)^2 = \log n,$$

which is not bounded. Now, by means of Theorem A and relations (2), (3), (4) and (5) it is easy to verify that

$$\underline{d}_{-1}(A) = \overline{d}_{-1}(A) = d_{-1}(A) = \frac{1}{2},$$

while, for every $\alpha > -1$, we have

$$\underline{d}_\alpha(A) = 0; \qquad \overline{d}_\alpha(A) = 1,$$

hence neither function $\alpha \mapsto \underline{d}_\alpha(A)$, $\alpha \mapsto \overline{d}_\alpha(A)$ is continuous in -1.

Theorem 2 covers evidently the rather relevant case of sets such the set E_r of numbers beginning by a fixed digit r ($r \in \{1, 2, \ldots, 9\}$), i. e.

$$E_r = \mathbb{N} \cap \Big(\bigcup_{n=1}^{\infty} \,]\, r10^n - 1, (r+1)10^n - 1] \Big),$$

but it is not useful for instance for the set of even numbers (or the set of multiples of any other integer, of course). In fact here we have $a_n = 2n - 1$, $b_n = 2n$ and

$$\liminf_{n\to\infty} (\log b_n - \log a_n) = 0.$$

Observe that in this case the limit

$$\lim_{n\to\infty} \frac{\log b_n - \log a_n}{\log b_n - \log b_{n-1}} = \lim_{n\to\infty} \frac{C_n}{B_n}$$

exists ($= 1/2$).

In fact, for a general set $A = \mathbb{N} \cap (\bigcup_{n=1}^{\infty} \,]\, a_n, b_n])$ the following result holds (we keep the notations used for Theorem 2):

Theorem 3. *Let assumption (H) hold and suppose that the limit*

$$\lim_{n\to\infty} \frac{C_n}{B_n}$$

exists and is equal to L. Put $b_0 = 1$. Then
(i) *A possesses logarithmic density $d_{-1}(A) = L$;*
(ii) *there exists α_0 and positive constant c such that for $-1 < \alpha < \alpha_0$ we have*

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} (b_k^{1+\alpha} - a_k^{1+\alpha})}{\sum_{k=1}^{n} (b_k^{1+\alpha} - b_{k-1}^{1+\alpha})} - \frac{\sum_{k=1}^{n} C_k}{\sum_{k=1}^{n} B_k} \right| \le c(1+\alpha). \tag{9}$$

As a consequence we get

$$\lim_{\alpha\to -1^+} \underline{d}_\alpha(A) = \lim_{\alpha\to -1^+} \overline{d}_\alpha(A) = d_{-1}(A).$$

4. An open problem

Problem. *We have seen that to any given set $A \subset \mathbb{N}$ we can attach a pair of functions*

$$\underline{d}_A : [-1, \infty[\to [0,1] \quad \textit{and} \quad \overline{d}_A : [-1, \infty[\to [0,1],$$

both continuous in the interval $]-1, \infty[$ such that $\underline{d}_A$ is nonincreasing, $\overline{d}_A$ is nondecreasing and $\underline{d}_A(\alpha) \le \overline{d}_A(\alpha)$ for all $\alpha \in [-1, \infty[$.
A natural question that arises is:
For which pairs of functions $\underline{d}, \overline{d}$ with properties listed above there exists a set $A \subset \mathbb{N}$ such that

$$\underline{d}_A = \underline{d} \quad \textit{and} \quad \overline{d}_A = \overline{d} \ ?$$

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