A Nitsche finite element method for dynamic contact

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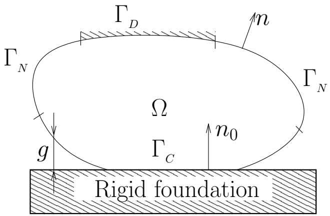
Setting: contact in linear elastodynamics



$\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$

Find $\mathbf{u} : [0, T) \times \Omega \to \mathbb{R}^d$ s.t.:

$$
\begin{aligned}
\rho \ddot{\mathbf{u}} - \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } (0, T) \times \Omega \\
\mathbf{u} &= \mathbf{0} && \text{on } (0, T) \times \Gamma_D \\
\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= \mathbf{g} && \text{on } (0, T) \times \Gamma_N \\
\mathbf{u}(0, \cdot) &= \mathbf{u_0} && \text{in } \Omega \\
\dot{\mathbf{u}}(0, \cdot) &= \dot{\mathbf{u}}_\mathbf{0} && \text{in } \Omega
\end{aligned}
$$

Contact without friction on $(0, T) \times \Gamma_C$:

$$
\begin{cases}
u_n \leq 0 \quad \sigma_n(\mathbf{u}) \leq 0 \quad \sigma_n(\mathbf{u})\, u_n = 0 & \text{(contact)} \\
\sigma_\mathbf{t}(\mathbf{u}) = 0 & \text{(no friction)}
\end{cases}
$$

with $\mathbf{u} = u_n \mathbf{n} + u_\mathbf{t}$, $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \sigma_\mathbf{t}(\mathbf{u})$.

Setting: contact in linear elastodynamics

Coulomb frictional case ($\sigma_{\mathbf{t}}(\mathbf{u}) \neq 0$) on $(0, T) \times \Gamma_C$:

$$\dot{\mathbf{u}}_{\mathbf{t}} = \mathbf{0} \implies \quad |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \leq -\mu\, \sigma_n(\mathbf{u})$$

$$\dot{\mathbf{u}}_{\mathbf{t}} \neq \mathbf{0} \implies \quad \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = \mu\, \sigma_n(\mathbf{u}) \frac{\dot{\mathbf{u}}_{\mathbf{t}}}{|\dot{\mathbf{u}}_{\mathbf{t}}|}$$

$\mu \geq 0$: friction coefficient ($\mu = 0 \leftrightarrow$ no friction)

Remark:

- No friction $\rightarrow$ energy conservation.
- Friction $\rightarrow$ energy dissipation when slip.

Theoretical results (without friction)

(Scalar) wave equation with unilateral constraints:

1. One dimensional case $\rightarrow$ one contact point:
   Existence and uniqueness.

   (Schatzman, 1980 ; Dabaghi, Petrov, Pousin, Renard, 2014)

2. $N \geq 2$ dimensional case:

   Existence and uniqueness (half space).
   (Lebeau, Schatzman, 1984)

   Existence (bounded domain with smooth boundary).
   (Kim, 1989)

Some well-known numerical difficulties

Discretization: method of lines.
Finite elements for space variables + time-marching schemes.

Contact condition: Lagrange multipliers or penalty.

Difficulties for the space semi-discrete problem:

1. ill–posedness with mixed FE discretization of contact conditions

   (Khenous, Laborde, Renard, 2008)
   $\rightarrow$ modified mass method

2. well–posedness of discrete systems with springs

   (Ballard, 2000 ; Ballard, Charles, 2014)

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Difficulties for the space-time discrete problem and for some conservative time-stepping (e.g., Crank-Nicolson).

1. Spurious oscillations on contact pressure, velocity and displacement.
2. No energy conservation and possible blow up.

(Khenous, Laborde, Renard, 2008 ; Doyen, Ern, Piperno, 2011 ; Krause, Walloth, 2012)

Some numerical contributions

(Non-exhaustive) list of strategies:

1. *"Modify the model"*
   Impact law. *(Paoli, 2001)*
2. *"Modify the time-discretization"*
   2.1 Implicit (dissipative) time-discretization of contact term. *(Carpenter-et-al, 1991 ; Kane-et-al, 1999 ; Dumont & Paoli, 2006 ; Deuflhard-et-al, 2008)*
   2.2 Velocity update method / contact condition with velocity. *(Laursen & Love, 2002 ; Laursen & Chawla, 1997)*
   2.3 Penalty with energy conservative schemes. *(Armero & Petöcz, 1998 ; Hauret & Le Tallec, 2006)*
3. *"Modify the space discretization"*
   Modified mass method and extensions.
   (Khenous, Laborde & Renard, 2008 ; Hager, Hüeber & Wohlmuth, 2008 ; Hauret, 2010 ; Renard, 2010)

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NFEM for dynamic contact

Key idea: contact condition can be reformulated as

$$\sigma_n(\mathbf{u}) = -\frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+$$

(Alart & Curnier, 1988)

Nitsche–FEM for (frictionless) dynamic contact:

$$\begin{cases} \text{Find } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \text{ s.t.:} \\ \langle \rho\ddot{\mathbf{u}}^h(t), \mathbf{v}^h\rangle + A_{\Theta\gamma_h}(\mathbf{u}^h(t), \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h}[P_{\gamma_h}(\mathbf{u}^h(t))]_+ P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma \\ \qquad = \quad L(t)(\mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}_0^h \qquad \dot{\mathbf{u}}^h(0,\cdot) = \dot{\mathbf{u}}_0^h \end{cases}$$

Notations:

- $\gamma_h := \gamma_0 h$
- $P_{\Theta\gamma_h}(\mathbf{v}^h) := v_n^h - \Theta\gamma_h\, \sigma_n(\mathbf{v}^h)$
- $A_{\Theta\gamma_h}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \displaystyle\int_{\Gamma_C} \Theta\gamma_h\, \sigma_n(\mathbf{u}^h)\sigma_n(\mathbf{v}^h)\, d\Gamma$

$\mathbf{V}^h$: H^1-conformal FE space ($\mathbb{P}_k$–cont, $k = 1, 2$).

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Two numerical parameters:

1. γ_0 (in $\gamma_h := \gamma_0 h$): **Nitsche's parameter.**
   Should be small enough for well-posedness
   (in the fully discrete framework).
2. Θ: **Select a variant.**
   Skew-symmetric (-1), non-symmetric (0) or symmetric (1).

(F.C., Hild & Renard, 2014)

Remark: consistency for all $\gamma_0 > 0$ ($\neq$ penalty).

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NFEM for dynamic contact : frictional case

No additional difficulty ...
since the frictional contact conditions can be rewritten:

$$
\begin{aligned}
\sigma_n(\mathbf{u}) &= -\frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+ \\
\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) &= -\frac{1}{\gamma}[\dot{\mathbf{u}}_{\mathbf{t}} - \gamma\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})]_{-\gamma\mu\sigma_n(\mathbf{u})} \\
&= -\frac{1}{\gamma}[\dot{\mathbf{u}}_{\mathbf{t}} - \gamma\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})]_{\mu[u_n-\gamma\sigma_n(\mathbf{u})]_+}
\end{aligned}
$$

Analysis of the semi-discrete problem. 1. Well-posedness

NFEM for dynamic contact can be recasted as:

$$\begin{cases} \text{Find } \mathbf{u}^h : [0, T] \to \mathbf{V}^h \text{ s.t.:} \\ \mathbf{M}^h \ddot{\mathbf{u}}^h(t) + \mathbf{B}^h \mathbf{u}^h(t) = \mathbf{L}^h(t) \\ \mathbf{u}^h(0, \cdot) = \mathbf{u}_0^h \qquad \dot{\mathbf{u}}^h(0, \cdot) = \dot{\mathbf{u}}_0^h \end{cases}$$

with

$$(\mathbf{M}^h \mathbf{v}^h, \mathbf{w}^h)_{\gamma_h} = \langle \rho \mathbf{v}^h, \mathbf{w}^h \rangle$$

$$(\mathbf{B}^h \mathbf{v}^h, \mathbf{w}^h)_{\gamma_h} = A_{\Theta\gamma_h}(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma_h} [P_{\gamma_h}(\mathbf{v}^h)]_+ P_{\Theta\gamma_h}(\mathbf{w}^h) \, d\Gamma$$

$$(\mathbf{L}^h(t), \mathbf{w}^h)_{\gamma_h} = L(t)(\mathbf{w}^h)$$

for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$, and
$(\mathbf{v}^h, \mathbf{w}^h)_{\gamma_h} := (\mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} + ({\gamma_h}^{-\frac{1}{2}} v_n^h, {\gamma_h}^{-\frac{1}{2}} w_n^h)_{0,\Gamma_C}$.

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Theorem (F.C., Hild, Renard)

$\mathbf{B}^h$ *is Lipschitz-continuous:*

$$\|\mathbf{B}^h \mathbf{v}_1^h - \mathbf{B}^h \mathbf{v}_2^h\|_{\gamma_h} \leq C(1 + \gamma_0)(1 + |\Theta|)\|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{\gamma_h}$$

for all $\mathbf{v}_1^h, \mathbf{v}_2^h \in \mathbf{V}^h$*;* $C > 0$ *independent of* h*,* Θ *and* γ_0*.*

So, for all $\Theta \in \mathbb{R}$ *and* $\gamma_0 > 0$*, Problem* $(\star)$ *admits one unique solution* $\mathbf{u}^h \in \mathscr{C}^2([0, T], \mathbf{V}^h)$*.*

Remark: the same result applies in the frictional case.

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Analysis of the semi-discrete problem.
2. Energy conservation

Proposition (F.C., Hild & Renard)

For a conservative system, i.e., $L(t) \equiv 0$ for all $t \in [0, T]$, the solution $\mathbf{u}^h$ of NFEM semi-discretization satisfies:

$$\frac{d}{dt} E_\Theta^h = (\Theta - 1) \int_{\Gamma_C} \frac{1}{\gamma_h} [P_{\gamma_h}(\mathbf{u}^h)]_+ \dot{u}_n^h \, d\Gamma$$

where

$$E_\Theta^h := E^h - \frac{\Theta}{2} \left[\| \gamma_h^{\frac{1}{2}} \sigma_n(\mathbf{u}^h) \|_{0,\Gamma_C}^2 - \| \gamma_h^{-\frac{1}{2}} [P_{\gamma_h}(\mathbf{u}^h)]_+ \|_{0,\Gamma_C}^2 \right]$$

$$E^h := \frac{1}{2} \rho \| \dot{\mathbf{u}}^h \|_{0,\Omega}^2 + \frac{1}{2} a(\mathbf{u}^h, \mathbf{u}^h)$$

When $\Theta = 1$, the augmented energy E_Θ^h is conserved.

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When $\Theta = 1$, the augmented energy E^h_Θ is conserved.

Time-marching schemes. 1. θ-scheme

Parameter is $\theta \in [0,1]$.
For $n \geq 0$, the fully discretized problem reads:

$$\begin{cases} \text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ such that:} \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau \dot{\mathbf{u}}^{h,n+\theta}, \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau \ddot{\mathbf{u}}^{h,n+\theta}, \\ \langle \rho \ddot{\mathbf{u}}^{h,n+1}, \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^{h,n+1}, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h} [P_{\gamma_h}(\mathbf{u}^{h,n+1})]_+ P_{\Theta\gamma_h}(\mathbf{v}^h) \, d\Gamma \\ = L^{n+1}(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

Well posedness when $\theta = 0$ or when $(1+\Theta)^2 \gamma_0 \leq C\left(1 + \frac{\rho h^2}{\tau^2 \theta^2}\right)$.
Stability (augmented energy) when $\Theta = 1$ (symmetric variant) and $\theta = 1$ (backward Euler).

Time-marching schemes. 2. Newmark scheme

Parameters are $\beta \in [0, 1/2]$, $\gamma \in [0, 1]$.
For $n \geq 0$, the fully discretized problem reads:

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Time-marching schemes: θ-scheme and Newmark

To summarize:

1. For the θ-scheme:
   numerical stability for arbitrary time-steps τ only for $\theta = 1$.

2. Newmark:
   numerical stability for arbitrary time-steps τ only for $\gamma = 1$, $\beta = \frac{1}{2}$.

3. Especially for **Crank-Nicolson**: no theoretical proof and no numerical evidence of (unconditional) stability.

4. Can we design a time-marching scheme that is stable and (almost) conservative ?

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Time-marching schemes. 3. A new hybrid scheme

Hybrid time-discretization of NFEM for dynamic contact:

$$\begin{cases} \text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ s.t.:} \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau \dot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau \ddot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \langle \rho \ddot{\mathbf{u}}^{h,n+\frac{1}{2}}, \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^{h,n+\frac{1}{2}}, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_h} \Phi(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1}) P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma = L^{n+\frac{1}{2}}(\mathbf{v}^h) \\ \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h \end{cases}$$

Initial conditions: $\mathbf{u}^{h,0} = \mathbf{u}_0^h,\ \dot{\mathbf{u}}^{h,0} = \dot{\mathbf{u}}_0^h,\ \ddot{\mathbf{u}}^{h,0} = \ddot{\mathbf{u}}_0^h.$

Hybrid Crank-Nicolson/Midpoint discretization of contact term:

$$\Phi(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1}) := H(P_{\gamma_h}(\mathbf{u}^{h,n}))[P_{\gamma_h}(\mathbf{u}^{h,n+\frac{1}{2}})]_+ + H(-P_{\gamma_h}(\mathbf{u}^{h,n}))[P_{\gamma_h}(\mathbf{u}^h)]_+^{n+\frac{1}{2}}$$

$H(\cdot)$ is the Heaviside function.

$\mathbf{x}^{h,n+\frac{1}{2}} = \frac{1}{2}\mathbf{x}^{h,n} + \frac{1}{2}\mathbf{x}^{h,n+1}$

(Gonzalez, 2000, Hauret & Le Tallec, 2006, F.C., Hild & Renard, 2015)

Time-marching schemes. 3. A new hybrid scheme

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*(Gonzalez, 2000, Hauret & Le Tallec, 2006,
F.C., Hild & Renard, 2015)*

Time-marching schemes. 3. A new hybrid scheme

Hybrid time-discretization of NFEM for dynamic contact:

$$(HN)\begin{cases} \text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ s.t.:} \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau \dot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau \ddot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \langle \rho \ddot{\mathbf{u}}^{h,n+\frac{1}{2}}, \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^{h,n+\frac{1}{2}}, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h} \Phi(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1}) P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma = L^{n+\frac{1}{2}}(\mathbf{v}^h) \\ \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h \end{cases}$$

Initial conditions: $\mathbf{u}^{h,0} = \mathbf{u}_0^h,\ \dot{\mathbf{u}}^{h,0} = \dot{\mathbf{u}}_0^h,\ \ddot{\mathbf{u}}^{h,0} = \ddot{\mathbf{u}}_0^h$.

Proposition (F.C., Hild, Renard)

If $(1+\Theta)^2 \gamma_0 \leq C\left(1 + \frac{\rho h^2}{\tau^2}\right)$, then for all n, Problem (HN) admits one unique solution.

Time-marching schemes. 3. A new hybrid scheme

Hybrid time-discretization of NFEM for dynamic contact:

$$(HN)\begin{cases} \text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ s.t.:} \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau \dot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau \ddot{\mathbf{u}}^{h,n+\frac{1}{2}} \\ \langle \rho \ddot{\mathbf{u}}^{h,n+\frac{1}{2}}, \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^{h,n+\frac{1}{2}}, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h} \Phi(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1}) P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma = L^{n+\frac{1}{2}}(\mathbf{v}^h) \\ \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h \end{cases}$$

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Time-marching schemes. 3. A new hybrid scheme

Proposition (F.C., Hild & Renard)

Suppose $L^n \equiv 0$ for all $n \geq 0$ and that Problem (HN) is well-posed. If γ_0 small (or $\Theta = -1$), then, for all $n \geq 0$:

$$
\begin{aligned}
&E_\Theta^{h,n+1} - E_\Theta^{h,n} \\
&= -\Theta \int_{\Gamma_C} \frac{1}{2\gamma_h} \big(H(P^n)H(P^n + P^{n+1})[P^{n+1}]_-^2 \\
&\quad + H(P^n)H(-P^n - P^{n+1})[P^n]_+^2 + [P^n]_-[P^{n+1}]_+\big)\ d\Gamma \\
&\quad + (\Theta - 1)\int_{\Gamma_C} \frac{1}{2\gamma_h}\big(H(P^n)[P^n + P^{n+1}]_+ + H(-P^n)\left([P^n]_+ + [P^{n+1}]_+\right)\big) \\
&\quad \left(u_n^{h,n+1} - u_n^{h,n}\right)\ d\Gamma
\end{aligned}
$$

with $P^n = P_{\gamma_h}(\mathbf{u}^{h,n})$, $P^{n+1} = P_{\gamma_h}(\mathbf{u}^{h,n+1})$ and

$$
E_\Theta^{h,n} := E^{h,n} - \frac{\Theta}{2}\left[\|\gamma_h^{\frac{1}{2}}\sigma_n(\mathbf{u}^{h,n})\|_{0,\Gamma_C}^2 - \|\gamma_h^{-\frac{1}{2}}[P_{\gamma_h}(\mathbf{u}^{h,n})]_+\|_{0,\Gamma_C}^2\right]
$$

$$
E^{h,n} := \frac{1}{2}\rho\|\dot{\mathbf{u}}^{h,n}\|_{0,\Omega}^2 + \frac{1}{2}a(\mathbf{u}^{h,n}, \mathbf{u}^{h,n})
$$

Time-marching schemes. 2. A new hybrid scheme

Corollary (F.C., Hild & Renard)

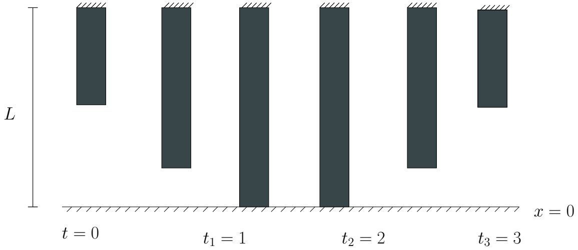
Suppose $L^n \equiv 0$ for all $n \geq 0$ and that Problem (HN) is well-posed.
If γ_0 small and $\Theta = 1$, then, for all $n \geq 0$:

$$
\begin{aligned}
& E_{\Theta}^{h,n+1} - E_{\Theta}^{h,n} \\
&= -\Theta \int_{\Gamma_C} \frac{1}{2\gamma_h} \Big(H(P^n)H(P^n + P^{n+1})[P^{n+1}]_-^2 \\
&\quad + H(P^n)H(-P^n - P^{n+1})[P^n]_+^2 + [P^n]_-[P^{n+1}]_+ \Big) \, d\Gamma \leq 0
\end{aligned}
$$

So the hybrid scheme is **unconditionally stable** *(for all $h, \tau > 0$).*

Numerical experiments. A 1D test-case

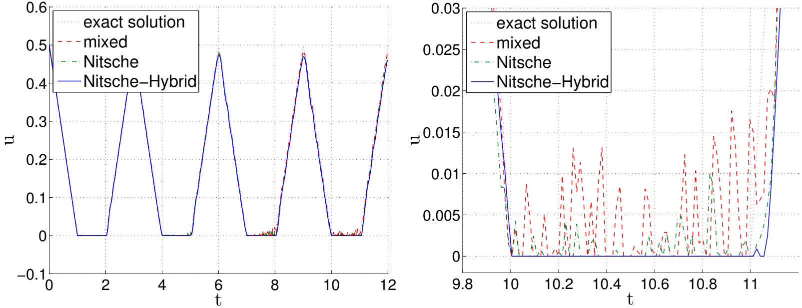
Multiple impacts of an elastic bar



- Parameters:
  $f = 0,\ E = 1,\ \rho = 1,\ L = 1,\ u_0(x) = \frac{1}{2} - \frac{x}{2},\ \dot{u}_0(x) = 0$
- Periodic analytical solution with multiple impacts.
  (Dabaghi-et-al, 2014)
- Numerical experiments with GetFEM++.
  `http://download.gna.org/getfem/html/homepage/`
- Generalized Newton to solve the non-linear problem.

Numerical experiments. A 1D test-case

Mixed(+Crank-Nicolson) vs. Nitsche(+Crank-Nicolson) vs. Nitsche-Hybrid



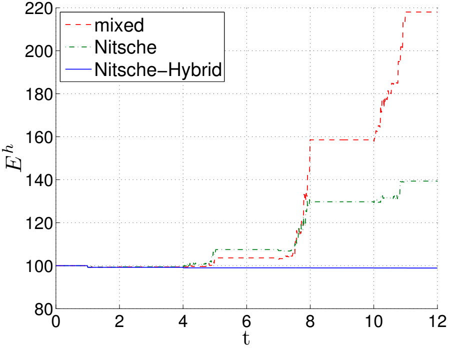
Displacement

Symmetric variant $\Theta = 1$, with $\gamma_0 = 10^{-6}$,
100 finite elements ($h = 0.01$), $\tau = 0.015$.

(F.C., Hild & Renard, 2015)

Numerical experiments. A 1D test-case

Mixed(+Crank-Nicolson) vs. Nitsche(+Crank-Nicolson) vs. Nitsche-Hybrid



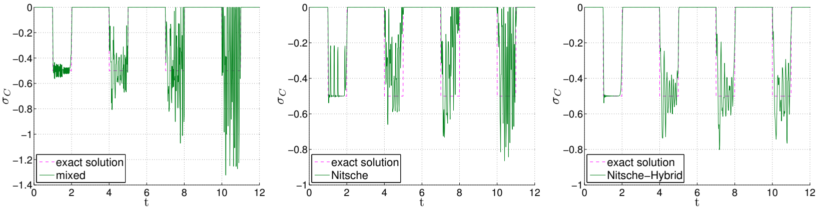
Energy

Symmetric variant $\Theta = 1$, with $\gamma_0 = 10^{-6}$,
100 finite elements ($h = 0.01$), $\tau = 0.015$.

(F.C., Hild & Renard, 2015)

Numerical experiments. A 1D test-case

Mixed(+Crank-Nicolson) vs. Nitsche(+Crank-Nicolson) vs. Nitsche-Hybrid



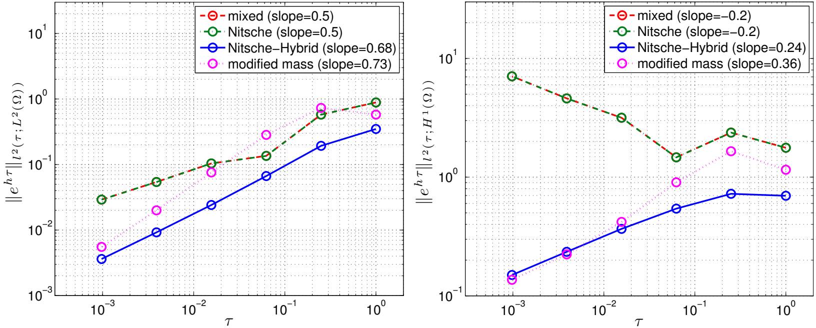
Contact pressure

Symmetric variant $\Theta = 1$, with $\gamma_0 = 10^{-6}$,
100 finite elements ($h = 0.01$), $\tau = 0.015$.

(F.C., Hild & Renard, 2015)

Numerical experiments. A 1D test-case

Convergence when $h, \tau \to 0$

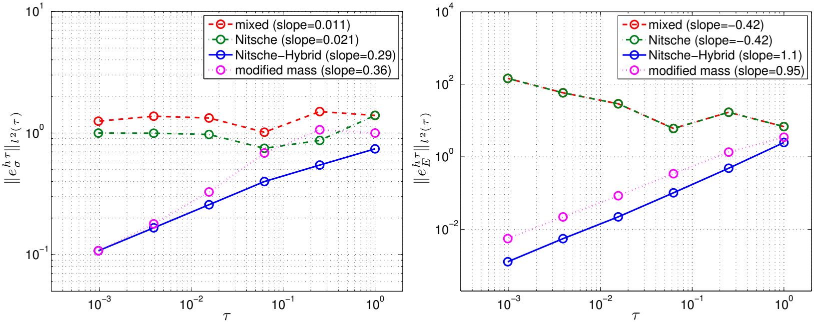


Error curves for u: error $e^{\tau h}$ in norm $l^2(\tau; L^2(\Omega))$ and $l^2(\tau; H^1(\Omega))$
Ratio τ/h is kept constant.

(F.C., Hild & Renard, 2015)

Numerical experiments. A 1D test-case

Convergence when $h, \tau \to 0$

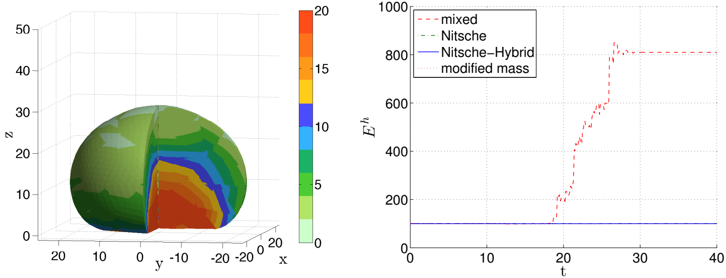


Error curves for the contact pressure σ_C and for the energy E.
Ratio τ/h is kept constant.

(F.C., Hild & Renard, 2015)

Numerical experiments. A 3D test-case

Impact of a ball



Deformed configuration and von Mises strain at $t = 18$, for Nitsche-Hybrid. Energy E^h for different methods.

Parameters: $D = 40$, $\lambda = 20$, $\mu = 20$, $\rho = 1$, $\|\mathbf{f}\| = 0.1$.
$\mathbb{P}_2$ Lagrange FEM, $h \simeq 8$ (400 elements), $\tau = 0.1$.
Symmetric variant $\Theta = 1$, with $\gamma_0 = 0.001$.

(F.C., Hild & Renard, 2015)

Conclusions

At the semi-discrete level: NFEM for dynamic contact

1. Consistency.
2. A well-posed semi-discrete problem.
3. Conservation of an augmented energy (when $\Theta = 1$) in the frictionless case.

Time-discretization of NFEM

1. Nitsche with Crank-Nicolson:
   still some spurious oscillations / no energy preservation.
2. Nitsche with the Hybrid scheme:
   no spurious oscillations on u + energy almost conserved.
3. Nitsche-Hybrid compares well to the modified mass method.

Further work

1. Theoretical study of convergence.
2. Schemes for frictional contact.

Conclusions

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