Inertial dynamical systems with vanishing damping. Fast algorithms for nonsmooth convex optimization.

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1a. Inertial dynamics, Asymptotic Vanishing Damping

$\mathcal{H}$ Hilbert, $\Phi : \mathcal{H} \to \mathbb{R}$ convex, differentiable, $\nabla\Phi$ loc Lip.

$$\min\left\{\Phi(x) : \quad x \in \mathcal{H}\right\}, \qquad S = \operatorname{argmin}\Phi \neq \emptyset.$$

Inertial dynamics with Asymptotic Vanishing Damping, $\alpha > 0$

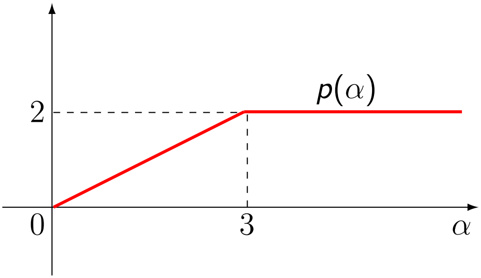
$$(\text{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

- $\alpha \geq 3$: Su-Boyd-Candès (NIPS, 2014):
  - $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(\frac{1}{t^2})$;
  - Link with Nesterov accelerated gradient method: $\alpha = 3$.
- $\alpha > 3$: A-Chbani-Peypouquet-Redont (Math Prog 16), May (15)
  - $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = o(\frac{1}{t^2})$;
  - $x(t) \rightharpoonup x_\infty \in S \quad as \quad t \to +\infty$.
- $\alpha < 3$: A-Chbani-Riahi (2017)
  - $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{t^{\frac{2\alpha}{3}}}\right)$

1b. Inertial dynamics, Asymptotic Vanishing Damping

$$(\mathrm{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{t^{p(\alpha)}}\right), \qquad p(\alpha) = \min\left(\frac{2\alpha}{3}, 2\right).$$



1c. Inertial dynamics & Optimization: brief history

$\gamma > 0$ fixed: Heavy Ball with Friction, Polyak (87)

$$\text{(HBF)} \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

- Convergence: Haraux-Jendoubi (98) analytic, Alvarez (2000) convex.
- $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(t^{-1})$.

$\gamma(t)$ vanishing: $\lim_{t \to +\infty} \gamma(t) = 0$.

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

- Cabot-Engler-Gadat (09)
  $$\int_{t_0}^{+\infty} \gamma(t)dt = +\infty \implies \Phi(x(t)) \to \min_{\mathcal{H}} \Phi.$$
- Cabot-Frankel (12), May (15),
  $$\gamma(t) = \frac{1}{t^\theta},\ 0 < \theta < 1 \Longrightarrow \Phi(x(t)) - \min_{\mathcal{H}} \Phi = o\left(\frac{1}{t^{1+\theta}}\right).$$
- Su-Boyd-Candès (14), A-Chbani-Peypouquet-Redont (16)
  $$\gamma(t) = \frac{\alpha}{t},\ \alpha \geq 3 \Longrightarrow \Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(t^{-2}).$$

2a. Inertial dynamics and Nesterov method.

$$(\text{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

Discretization: explicit /smooth Φ.

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{kh^2}(x_k - x_{k-1}) + \nabla\Phi(y_k) = 0.$$

$$\Updownarrow$$

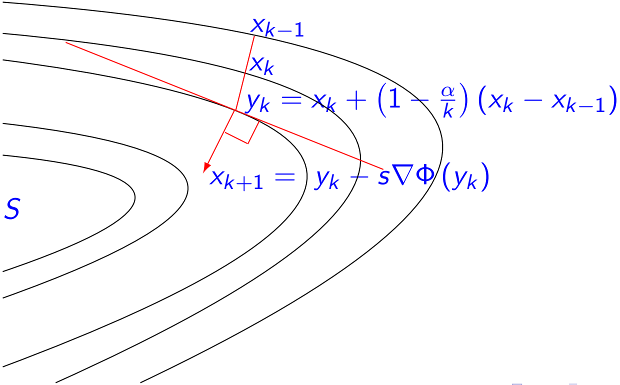
$$x_{k+1} = \left(x_k + \left(1 - \tfrac{\alpha}{k}\right)(x_k - x_{k-1})\right) - h^2\nabla\Phi(y_k).$$

Classical choice (Nesterov): $y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}), \; s = h^2$

$$(\text{IG})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}); \\ x_{k+1} = y_k - s\nabla\Phi(y_k). \end{cases}$$

Accelerated gradient method: Φ convex differentiable.

$$(\mathrm{IG})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = y_k - s\nabla\Phi(y_k) \end{cases}$$



2c. Inertial dynamics and Nesterov method.

$$(\mathrm{IG})_\alpha \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = y_k - s\nabla\Phi(y_k) \end{cases}$$

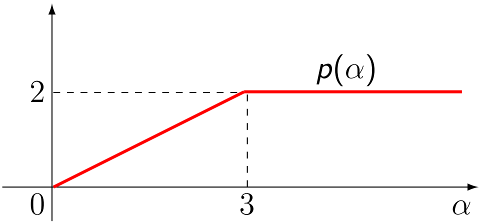
- $\alpha = 3$: Nesterov (1983, 2007)
  - $\Phi(x_k) - \min_{\mathcal{H}} \Phi = \mathcal{O}(\frac{1}{k^2})$;
  - Convergence of (x_k): open question.
- $\alpha > 3$: Chambolle-Dossal (2015), Attouch-Peypouquet (2016)
  - $\Phi(x_k) - \min_{\mathcal{H}} \Phi = o(\frac{1}{k^2})$;
  - $x_k \rightharpoonup \bar{x} \in S$: weak convergence.
- $\alpha \leq 3$: A-Chbani-Riahi (2017)
  - $\Phi(x_k) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{k^{\frac{2\alpha}{3}}}\right)$;

2d. Inertial dynamics and Nesterov method.

$$(\mathrm{IG})_\alpha \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = y_k - s\nabla\Phi(y_k) \end{cases}$$

$$\Phi(x_k) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{k^{p(\alpha)}}\right), \qquad p(\alpha) = \min\left(\frac{2\alpha}{3}, 2\right).$$

First-order methods: $\Phi(x_k) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{k^2}\right)$ worst case optimal.



3a. Sketch of the proof. Lyapunov analysis.

$$(\mathrm{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

Theorem 1 (SBC, ACPR, ACR)

Let $\Phi : \mathcal{H} \to \mathbb{R}$ *be a convex continuously differentiable function such that* $\operatorname{argmin}\Phi \neq \emptyset$. *Let* $x(\cdot)$ *be a classical global solution of* $(\mathrm{AVD})_\alpha$.

- $\alpha \geq 3$: $\quad \Phi(x(t)) - \min_{\mathcal{H}}\Phi = \mathcal{O}(\frac{1}{t^2})$.
- $0 < \alpha \leq 3$: $\quad \Phi(x(t)) - \min_{\mathcal{H}}\Phi = \mathcal{O}(\frac{1}{t^{\frac{2\alpha}{3}}})$.

Lyapunov function, $z \in \operatorname{argmin}\Phi$, $p > 0$, $\lambda(\cdot)$, $\xi(\cdot)$ parameters

$$\mathcal{E}(t) = t^{2p}\left[\Phi(x(t)) - \min_{\mathcal{H}}\Phi\right] + \frac{1}{2}\|\lambda(t)(x(t)-z) + t^p\dot{x}(t)\|^2 + \frac{\xi(t)}{2}\|x(t)-z\|^2$$

$$= t^{2p}W(t) + (\lambda(t)^2 + \xi(t))h(t) + \lambda(t)t^p\dot{h}(t)$$

3b. Proof of the convergence rate.

$$\mathcal{E}(t) = t^{2p}\left[\Phi(x(t)) - \min_{\mathcal{H}} \Phi\right] + \frac{1}{2}\|\lambda(t)(x(t)-z) + t^p \dot{x}(t)\|^2 + \frac{\xi(t)}{2}\|x(t)-z\|^2$$

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) \leq t^p\left[2pt^{p-1} - \lambda(t))\right](\Phi(x(t)) - \min_{\mathcal{H}}\Phi) \\
&+ \left[\xi(t) + t^p\dot{\lambda}(t) - (\alpha - p)t^{p-1}\lambda(t) + \lambda(t)^2\right]\langle x(t)-z, \dot{x}(t)\rangle \\
&- t^p\left[(\alpha-p)t^{p-1} - \lambda(t)\right]\|\dot{x}(t)\|^2 + \left[\lambda(t)\dot{\lambda}(t) + \frac{\dot{\xi}(t)}{2}\right]\|x(t)-z\|^2.
\end{aligned}$$

(H$_1$): $\lambda(t) = 2pt^{p-1}$ $\Rightarrow$ first term equal to zero.

(H$_2$): $\xi(t) = 2(\alpha - 4p + 1)pt^{2(p-1)}$ $\Rightarrow$ second term equal to zero.

(H$_3$): $\alpha + 1 \geq 4p \Rightarrow \xi \geq 0$, and hence $\mathcal{E} \geq 0$.

(H$_4$): $\alpha \geq 3p \Rightarrow$ third term less or equal than zero.

(H$_5$): $1 \geq p \Rightarrow$ fourth term less or equal than zero.

Take $p = \min(1, \frac{\alpha}{3}, \frac{\alpha+1}{4}) = \min(1, \frac{\alpha}{3})$.

4. Case of a strong minimum

The convex function Φ has a strong minimum $x^* \in \mathcal{H}$, if

$$\exists \eta > 0 \; s.t. \; \Phi(x) \geq \Phi(x^*) + \frac{\eta}{2}\|x - x^*\|^2 \quad \forall x \in \mathcal{H}.$$

The convergence rate increases indefinitely with larger values of α.

Theorem 2 (SBC, ACPR, AC)

Suppose that the convex function $\Phi : \mathcal{H} \to \mathbb{R}$ has a strong minimum $x^* \in \mathcal{H}$, and let $x : [t_0, +\infty[\to \mathcal{H}$ be a solution of $(\mathrm{AVD})_\alpha$ with $\alpha > 0$. Then $x(t)$ converges strongly to x^* the unique element of $\operatorname{argmin} \Phi$, and

(i) $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{t^{\frac{2\alpha}{3}}}\right)$

(ii) $\|x(t) - x^*\|^2 = \mathcal{O}\left(t^{-\frac{2}{3}\alpha}\right)$,

(iii) $\|\dot{x}(t)\| = \mathcal{O}(t^{-\frac{1}{3}\alpha})$.

4b. Example $\Phi(x) = \frac{1}{2}\|x\|^2$. Role of Bessel functions

$$(\text{AVD})_\alpha \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + x(t) = 0.$$

Solution of $(\text{AVD})_\alpha$ with Cauchy data $x(0) = x_0$, $\dot{x}(0) = 0$:

$$x(t) = 2^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha+1}{2})\frac{J_{\frac{\alpha-1}{2}}(t)}{t^{\frac{\alpha-1}{2}}}x_0.$$

$J_{\frac{\alpha-1}{2}}(\cdot)$: first kind Bessel function of order $\frac{\alpha-1}{2}$. For large t,

$$J_\alpha(t) = \sqrt{\frac{2}{\pi t}}\left(\cos\left(t - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\tfrac{1}{t})\right).$$

Hence

$$\Phi(x(t)) - \min_{\mathcal{H}}\Phi = \mathcal{O}(t^{-\alpha}).$$

Compare with $\mathcal{O}(t^{-\frac{2}{3}\alpha})$, valid for arbitrary strongly convex functions.

5a. Nonsmooth structured optimization

Nonsmooth structured convex minimization.

$$\min\{\Phi(x)+\Psi(x): \quad x\in\mathcal{H}\}.$$

- $\mathcal{H}$ real Hilbert space.
- $\Phi:\mathcal{H}\to\mathbb{R}$ convex, differentiable, $\nabla\Phi$ L-Lipschitz continuous.
- $\Psi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ convex, lower semicontinuous, proper.
- $S=\operatorname{argmin}(\Phi+\Psi)\neq\emptyset$.

Model examples.

- Sparse optimization:

$$\min\left\{\|Ax-b\|_2^2+\lambda\|x\|_1: \quad x\in\mathbb{R}^n\right\}.$$

- Constrained optimization ($\delta_C=0$ on C, $+\infty$ elsewhere):

$$\min\{\Phi(x)+\delta_C(x): \quad x\in\mathcal{H}\}.$$

5b. Fast splitting algorithms

Structured convex minimization.

$$\min \{\Phi(x) + \Psi(x) : \quad x \in \mathcal{H}\}.$$

- $\Phi : \mathcal{H} \to \mathbb{R}$ convex, differentiable, $\nabla\Phi$ Lipschitz continuous.
- $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lower semicontinuous.

Splitting algorithms. Basic blocks:

- Gradient (forward, explicit) w.r. smooth function Φ.
- Proximal (backward, implicit) w.r. nonsmooth function Ψ.

Combine $\rightsquigarrow$ Proximal-Gradient, Forward-Backward algorithms.

Inertial dynamics with vanishing damping $\rightsquigarrow$ fast algorithms.

- Fast minimization (in the worst case):

$$(\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) = o\left(\frac{1}{k^2}\right).$$

5c. Classical Proximal-Gradient: Gradient methods

$\Phi : \mathcal{H} \to \mathbb{R}$ convex, $\mathcal{C}^1$, $\nabla\Phi$ L-Lipschitz, $S = \text{argmin}\Phi \neq \emptyset$.

$$\min \{\Phi(x) : \quad x \in \mathcal{H}\}.$$

Gradient flow (continuous steepest descent).

$$\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

Brézis, Bruck, Baillon $(70 - 80)$: $x(t) \rightharpoonup x_\infty \in S$ weakly as $t \to +\infty$.

Time discretization (explicit, forward): Gradient algorithm.

- $\frac{1}{s}(x_{k+1} - x_k) + \nabla\Phi(x_k) = 0 \iff x_{k+1} = x_k - s\nabla\Phi(x_k)$.
- $0 < s < \frac{2}{L}$: $x_k \rightharpoonup x_\infty \in S$ weakly, $\Phi(x_k) - \min_{\mathcal{H}} \Phi = \mathcal{O}(\frac{1}{k})$.

5d. Classical Proximal-Gradient: Proximal methods

$\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ closed, convex, proper, $S = \text{argmin}\Psi \neq \emptyset$.

$$\min \{\Psi(x) : \quad x \in \mathcal{H}\}.$$

Differential inclusion governed by $\partial\Psi : \mathcal{H} \rightrightarrows \mathcal{H}$ subdifferential

$$\dot{x}(t) + \partial\Psi(x(t)) \ni 0.$$

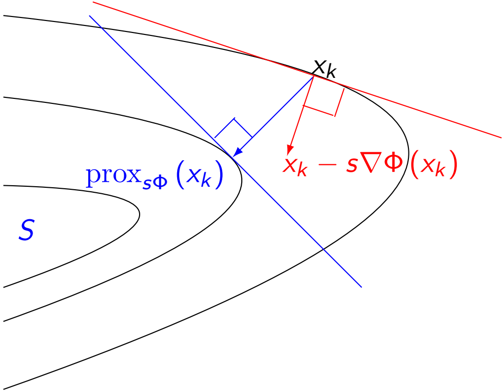
Brézis, Bruck, Baillon $(70 - 80)$: $x(t) \rightharpoonup x_\infty \in S$ weakly as $t \to +\infty$.

Time discretization (implicit, backward): Proximal algorithm.

- $\frac{1}{s}(x_{k+1} - x_k) + \partial\Psi(x_{k+1}) \ni 0 \iff x_{k+1} = \text{prox}_{s\Psi}(x_k)$.
- $\text{prox}_{s\Psi}(x) = \text{armin}_{\xi\in\mathcal{H}} \left\{s\Psi(\xi) + \frac{1}{2}\|x - \xi\|^2\right\} = (I + s\partial\Psi)^{-1}(x)$.
- $s > 0$, $x_k \rightharpoonup x_\infty \in S$ weakly, $\Psi(x_k) - \min_{\mathcal{H}} \Psi = \mathcal{O}(\frac{1}{k})$.

$$\Phi : \mathcal{H} \to \mathbb{R} \text{ convex, } \mathcal{C}^1.$$

Gradient: $x_{k+1} = x_k - s\nabla\Phi(x_k)$ Proximal: $x_{k+1} = \text{prox}_{s\Phi}(x_k)$



5f. Classical Proximal-Gradient algorithm

$$\min\left\{\Phi(x)+\Psi(x): \quad x\in\mathcal{H}\right\}.$$

- $\Phi:\mathcal{H}\to\mathbb{R}$ convex, $\mathcal{C}^1$, $\nabla\Phi$ L-Lipschitz continuous.
- $\Psi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ proper, convex, lower semicontinuous.

Differential inclusion with two potentials.

$$\dot{x}(t)+\nabla\Phi(x(t))+\partial\Psi(x(t))\ni 0.$$

Finite difference discretization: step $s>0$, $t_k=ks$, $x_k=x(t_k)$.

Explicit /smooth function Φ. Implicit /nonsmooth function Ψ.

$$s^{-1}(x_{k+1}-x_k)+\nabla\Phi(x_k)+\partial\Psi(x_{k+1})\ni 0;$$

$$\Updownarrow$$

$$x_{k+1}=\operatorname{prox}_{s\Psi}\left(x_k-s\nabla\Phi(x_k)\right).$$

5g. Classical Proximal-Gradient algorithm

Goldstein (64), Levitin-Polyak (66), Lions-Mercier (79), Passty (79),...

- $\Phi : \mathcal{H} \to \mathbb{R}$ convex, $\mathcal{C}^1$, $\nabla\Phi$ L-Lipschitz continuous.
- $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lower semicontinuous.
- $0 < s < \frac{2}{L}$.

$$x_{k+1} = \text{prox}_{s\Psi}\left(x_k - s\nabla\Phi(x_k)\right).$$

- $i)$ $(\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) = \mathcal{O}(\frac{1}{k})$;
- $ii)$ $x_k \rightharpoonup \bar{x} \in S$.

Examples

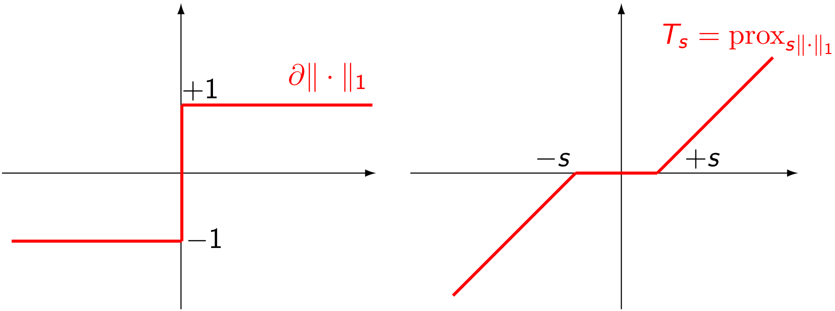
- $\Psi = \delta_C$: Gradient-Projection, $x_{k+1} = \text{proj}_C\left(x_k - s\nabla\Phi(x_k)\right)$.
- $\Psi = \|\cdot\|_1$: ISTA, Iterative Soft Thresholding Algorithm.

5h. Classical Proximal-Gradient algorithm: ISTA

ISTA: $\Psi = \lambda \| \cdot \|_1$, Iterative Soft Thresholding Algorithm.

$$\min \left\{ \Phi(x) + \lambda \|x\|_1 : \quad x \in \mathbb{R}^n \right\}.$$

$$x_{k+1} = T_s \left(x_k - \frac{s}{\lambda} \nabla \Phi(x_k) \right), \quad 0 < s < \frac{2\lambda}{L}.$$



6a. Inertial dynamics: nonsmooth potential.

Non-smooth structured convex minimization

$$\min\{\Phi(x)+\Psi(x):\ x\in\mathcal{H}\}.$$

- $\Phi:\mathcal{H}\to\mathbb{R}$ convex, differentiable, $\nabla\Phi$ Lipschitz continuous.
- $\Psi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ closed, convex, proper;

Dynamic approach via the differential inclusion

$$\ddot{x}(t)+\frac{\alpha}{t}\dot{x}(t)+\nabla\Phi(x(t))+\partial\Psi(x(t))\ni 0.$$

- Unilateral mechanics, shocks: Paoli-Schatzman, A-Cabot-Redont. x loc. Lipschitz; $\dot{x}$ bounded variation; $\ddot{x}$ bounded measure.
- Lyapunov analysis is still valid: convex subdifferential inequalities, generalized derivation chain rule.

6b. Link with inertial dynamics: algorithms.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) + \partial\Psi(x(t)) \ni 0.$$

Discretization: explicit /smooth Φ, implicit /nonsmooth Ψ.

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{kh^2}(x_k - x_{k-1}) + \nabla\Phi(y_k) + \partial\Psi(x_{k+1}) \ni 0.$$

$$\Updownarrow$$

$$x_{k+1} + h^2\partial\Psi(x_{k+1}) \ni \left(x_k + \left(1 - \tfrac{\alpha}{k}\right)(x_k - x_{k-1})\right) - h^2\nabla\Phi(y_k).$$

Classical choice (Nesterov): $y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1})$, $s = h^2$

$$(\mathrm{IPG})_\alpha \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}); \\ x_{k+1} = \mathrm{prox}_{s\Psi}\left(y_k - s\nabla\Phi(y_k)\right). \end{cases}$$

6c. Inertial Proximal-Gradient algorithm: $(\text{IPG})_\alpha$

Structured convex minimization. $\mathcal{H}$ real Hilbert space

$$\min\left\{\Phi(x)+\Psi(x): \quad x\in\mathcal{H}\right\}.$$

- $\Phi:\mathcal{H}\to\mathbb{R}$ convex, $\mathcal{C}^1$, $\nabla\Phi$ L-Lipschitz continuous.
- $\Psi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ convex, lower semicontinuous, proper.
- $S=\operatorname{argmin}(\Phi+\Psi)\neq\emptyset$.

Theorem 3 (Chambolle-Dossal; A-Peypouquet; A.-Chbani-Riahi) $0<s\leq\frac{1}{L}$

$$(\text{IPG})_\alpha \quad \begin{cases} y_k &= x_k+\left(1-\frac{\alpha}{k}\right)(x_k-x_{k-1}) \\ x_{k+1} &= \operatorname{prox}_{s\Psi}\left(y_k-s\nabla\Phi(y_k)\right)\end{cases}$$

$\alpha>3$: $(\Phi+\Psi)(x_k)-\min_{\mathcal{H}}(\Phi+\Psi)=o\left(\frac{1}{k^2}\right)$; $x_k\rightharpoonup\bar{x}\in S$.

$\alpha\leq 3$: $(\Phi+\Psi)(x_k)-\min_{\mathcal{H}}(\Phi+\Psi)=\mathcal{O}\left(\frac{1}{k^{\frac{2\alpha}{3}}}\right)$;

6d. $(\text{IPG})_\alpha$ algorithm and FISTA

$\alpha = 3$: Nesterov (1983, 2007), Beck-Teboulle (2009)

$$(\text{FISTA}) \begin{cases} y_k &= x_k + \left(1 - \frac{3}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} &= \text{prox}_{s\Psi}\left(y_k - s\nabla\Phi(y_k)\right) \end{cases}$$

- $(\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) = \mathcal{O}(\frac{1}{k^2})$;
- Convergence of (x_k): open question.

$\alpha > 3$: SBC (2014), CD 2015), AP(2016), ACR(2017)

.

$$(\text{IPG})_\alpha \begin{cases} y_k &= x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} &= \text{prox}_{s\Psi}\left(y_k - s\nabla\Phi(y_k)\right) \end{cases}$$

- $(\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) = o(\frac{1}{k^2})$;
- $x_k \rightharpoonup \bar{x} \in S$: weak convergence.

7. Newton-like inertial dynamic with Hessian damping

$$\text{(DIN-AVD)}_{\alpha,\beta} \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

$\text{(DIN-AVD)}_{\alpha,\beta}$ looks much more complicated, but

Proposition (A-Peypouquet-Redont, JDE 2016)

$\text{(DIN-AVD)}_{\alpha,\beta}$ is equivalent to

$$\begin{cases} \dot{x}(t) + \beta\nabla\Phi(x(t)) - \left(\frac{1}{\beta} - \frac{\alpha}{t}\right)x(t) + \frac{1}{\beta}y(t) = 0; \\ \dot{y}(t) - \left(\frac{1}{\beta} - \frac{\alpha}{t} + \frac{\alpha\beta}{t^2}\right)x(t) + \frac{1}{\beta}y(t) = 0, \end{cases}$$

- First order system in time and space.
- In the product space: linear perturbation of a gradient system.
- Nonsmooth setting: similar results (damped shocks in mechanics).
- Time discretization gives inertial Newton-like algorithms.

8. Adaptive restart (SBC)

Strategy: maintain high velocity along the orbit.

$$(\mathrm{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0, \quad x(0) = x_0, \ \dot{x}(0) = 0.$$

Restarting time: $T(\Phi, x_0) = \sup\{t > 0, \ \forall \tau \in]0, t[, \ \frac{d}{d\tau}\|\dot{x}(\tau)\|^2 > 0\}$.

Before time $T(\Phi, x_0) > 0$, $t \mapsto \Phi(x(t))$ decreases:

$$\frac{d}{dt}\Phi(x(t)) = \langle \nabla\Phi(x(t)), \dot{x}(t)\rangle = -\frac{\alpha}{t}\|\dot{x}(t)\|^2 - \frac{1}{2}\frac{d}{dt}\|\dot{x}(t)\|^2 \leq 0.$$

At time $T(\Phi, x_0)$, stop and restart, and so on.

Theorem (SBC), linear convergence

Suppose $\Phi : \mathcal{H} \to \mathbb{R}$ strongly convex, $\nabla\Phi$ Lipschitz continuous, $\alpha \geq 3$. Let $x_{sr}(\cdot)$ be an orbit of the speed restarting dynamic. Then

$$\Phi(x_{sr}(t)) - \min_{\mathcal{H}} \Phi \leq c_1 e^{-c_2 t}.$$

9. Nesterov acceleration with Tikhonov regularization.

$$(\text{AVD})_{\alpha,\epsilon} \qquad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) + \epsilon(t)x(t) = 0,$$

Theorem (A-Chbani, JMAA 2016; critical size $\epsilon(t) = \frac{1}{t^2}$)

- Fast vanishing: $\epsilon(t) = \frac{1}{t^r}$, $r > 2$, $\alpha > 3$

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi = o(t^{-2}); \quad x(t) \rightharpoonup x_\infty \in \text{argmin}_{\mathcal{H}}\Phi \ \text{ weakly.}$$

- Slow: $\epsilon(t) = \frac{1}{t^r}$, $0 < r < 2$, or $\epsilon(t) = \frac{c}{t^2}$, $\alpha > 3$, $c > \frac{4}{9}\alpha(\alpha - 3)$.

$$\liminf_{t\to\infty} \|x(t) - x^*\| = 0, \text{ where } x^* = \text{proj}_{\text{argmin}\Phi} 0.$$

- Very slow vanishing: $\epsilon(t) = \frac{1}{(\ln t)^\gamma}$, $0 < \gamma \le 1$:

$$\lim_{t\to+\infty} \frac{1}{\int_{t_0}^t \frac{\epsilon(\tau)}{\tau} d\tau} \int_{t_0}^t \frac{\epsilon(\tau)}{\tau} \|x(\tau) - x^*\| d\tau = 0.$$

10. Perspective, open questions

- $\alpha = 3$ critical?
  - Convergence of the orbits for Nesterov algorithm ($\alpha = 3$)?
  - Convergence of the values: exhibit concrete examples showing that $\alpha = 3$ is critical.
  - Numerical evidence taking α large. Variant Liang-Fadili-Peyré.
- $(\text{DIN-AVD})_{\alpha,\beta}$.
  - Algorithmic version.
  - Adaptive restart for $(\text{DIN-AVD})_{\alpha,\beta}$, without strong convexity.
- Non-convex setting, semialgebraic, Kurdyka-Lojasiewicz. For analytic potentials, the convergence theory for HBF (Haraux-Jendoubi, Bolte et al.), and DIN (AABR) still works.
- Combine Nesterov and Tikhonov.
  - Obtening both strong convergence and fast convergence.
  - Accelerate Haugazeau method (closed-loop Tikhonov method).
- Develop $(\text{AVD})_{\alpha}$ for other (splitting) methods, for maximal monotone operators, primal-dual methods.

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