A CLASS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES IN CONTACT MECHANICS

Mircea Sofonea

Laboratoire de Mathématiques et Physique,
Université de Perpignan Via Domitia,
52 Avenue Paul Alduy,
66860 Perpignan, France

FBP17, Saint-Etienne, France, June 2017

CONTENTS

I. ONE-DIMENSIONAL EXAMPLE

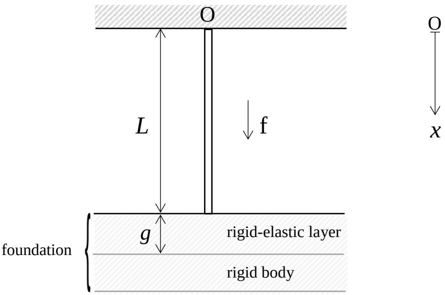

Figure: 1. Physical setting.

Problem $\mathcal{P}^{1d}$. *Find a displacement field* $u\colon [0,L] \to \mathbb{R}$ *and a stress field* $\sigma\colon [0,L] \to \mathbb{R}$ *such that*

$$\sigma(x) = \mathcal{F}\, u'(x) \quad \text{for } x \in (0,L),$$

$$\sigma'(x) + f(x) = 0 \quad \text{for } x \in (0,L),$$

$$u(0) = 0,$$

$$u(L) \le g, \qquad \left.\begin{array}{ll} \sigma(L) = 0 & \text{if } \ u(L) < 0 \\ -\sigma(L) \in [0,P] & \text{if } u(L) = 0 \\ -\sigma(L) = P + p(u(L)) & \text{if } \ 0 < u(L) < g \\ -\sigma(L) \ge P + p(u(L)) & \text{if } \ u(L) = g \end{array}\right\} .$$

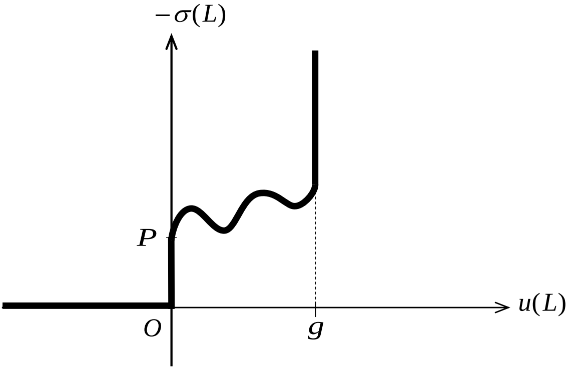

Figure: 2. The contact conditions.

We have four posibilities:

a) $u(L) < 0 \Longrightarrow \sigma(L) = 0$: separation, no reaction on $x = L$.

b) $u(L) = 0 \Longrightarrow -\sigma(L) \in [0, P]$: contact, reaction towards the rod, no penetration.

c) $0 < u(L) < g \Longrightarrow -\sigma(L) = P + p(u(L))$: partial penetration.

d) $u(L) = g \Longrightarrow -\sigma(L) \geq P + p(u(L))$: the rigid-elastic layer is completely penetrated; the reaction of the rigid body is active.

Denote:

$V = \{ v \in H^1(0,L) \mid v(0) = 0 \}$,

$(u,v)_V = \int_0^L u'\, v'\, dx \qquad \forall\ u,\ v \in V$,

$\| \cdot \|_V$ - the associated norm,

V^* - the dual of V,

$\langle \cdot, \cdot \rangle$ - the duality pairing between V^* and V.

$$q \colon \mathbb{R} \to \mathbb{R}, \quad q(r) = \int_0^r p(s)\, ds \qquad \text{for all} \quad r \in \mathbb{R},$$

$$K_g = \{\, u \in V \mid u(L) \le g \,\},$$

$$A\colon V \to V^*, \quad \langle Au, v\rangle = \int_0^L \mathcal{F}(u')\, v'\, dx \qquad \text{for all} \;\; u, v \in V,$$

$$\pi\colon V \to L^2(0,L), \quad \pi v = v \qquad \text{for all} \;\; v \in V \;,$$

$$\varphi\colon V \to \mathbb{R}, \quad \varphi(v) = Pv(L)^+ \qquad \text{for all} \;\; v \in V,$$

$$j\colon V \to \mathbb{R}, \quad j(v) = q(v(L)) \qquad \text{for all} \;\; v \in V.$$

Problem $\mathcal{P}_V^{1d}$. *Find a displacement field $u \in K_g$ such that*

$$\langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \ge (f, \pi v - \pi u)_{L^2(0,L)} \quad \forall\, v \in K_g.$$

II. EXISTENCE AND UNIQUENESS

Problem $\mathcal{P}$. *Given $f \in Y$ and $g > 0$, find $u \in K_g$ such that*

$$\langle Au, v-u\rangle+\varphi(u,v)-\varphi(u,u)+j^0(u;v-u) \geq (f, \pi v-\pi u)_Y \quad \forall\, v \in K_g.$$

Here:

X - reflexive Banach space of dual X^*,

$K \subset X$, $K_g = gK$,

$A : X \to X^*$, $\varphi : X \times X \to \mathbb{R}$, $j : X \to \mathbb{R}$,

Y- Hilbert space with the inner product $(\cdot,\cdot)_Y$,

$\pi : X \to Y$.

Assumptions

(1) K is a nonempty, closed and convex subset of X.

$$(2)\ \begin{cases} A: X \to X^* \text{ is strongly monotone and Lipschitz} \\ \text{continuous, i.e.,} \\ \langle Au - Av, u - v\rangle \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X \text{ with } m_A > 0, \\ \|Au - Av\|_{X^*} \leq L_A \|u - v\|_X \quad \forall\, u, v \in X \text{ with } L_A > 0 \end{cases}$$

$$(3)\ \begin{cases} \pi: X \to Y \text{ is a linear continuous operator, i.e.,} \\ \|\pi v\|_Y \leq d_0 \|v\|_X \quad \forall\, v \in X \text{ with } d_0 > 0. \end{cases}$$

$$
(4)\quad \begin{cases}
\text{(a) } \varphi(\eta,\cdot)\colon X \to \mathbb{R} \text{ is convex and l.s.c.,} \\
\qquad\quad \text{for all } \eta \in X. \\
\text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\
\quad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\
\qquad \le \alpha_\varphi \|\eta_1 - \eta_2\|_X \, \|v_1 - v_2\|_X \\
\qquad\quad \text{for all } \eta_1,\ \eta_2 \in X, v_1,\ v_2 \in X.
\end{cases}
$$

$$
(5)\quad \begin{cases}
\text{(a) } j\colon X \to \mathbb{R} \text{ is locally Lipschitz.} \\
\text{(b) } \|\partial j(v)\|_{X^*} \le c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \ge 0. \\
\text{(c) there exists } \alpha_j > 0 \text{ such that} \\
\qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \\
\qquad\quad \text{for all } v_1, v_2 \in X.
\end{cases}
$$

Theorem 1. *Assume that* (1)–(5) *and, in addition, assume that*

$$\alpha_\varphi + \alpha_j < m_A. \tag{6}$$

Then, for each $f \in Y$ and $g > 0$, Problem $\mathcal{P}$ has a unique solution $u = u(f, g)$.

Proof. We use an existence and uniqueness result proved in

S. Migórski, A. Ochal & **M. Sofonea**, A Class of Variational-Hemivariational Inequalities in Reflexive Banach Spaces, *Journal of Elasticity* **127** (2017), 151–178.

III. A CONVERGENCE RESULT

Assume that

$$(7) \qquad 0_X \in K.$$

$$(8) \quad \left\{ \begin{array}{l} \varphi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a)} \;\; \varphi(u, \lambda v) = \lambda \varphi(u, v) \qquad \forall\, u,\, v \in X,\;\; \lambda > 0. \\ \text{(b)} \;\; \varphi(v, v) \geq 0 \quad \forall\, u,\, v \in X. \\ \text{(c)} \; \eta_n \rightharpoonup \eta \quad \text{in} \quad X,\;\; u_n \rightharpoonup u \quad \text{in} \quad X \quad \Longrightarrow \\ \qquad \limsup \left[\varphi(\eta_n, v) - \varphi(\eta_n, u_n)\right] \leq \varphi(\eta, v) - \varphi(\eta, u) \\ \qquad \forall\, v \in X. \end{array} \right.$$

$$(9) \quad \left\{ \begin{array}{l} j\colon X \to \mathbb{R} \text{ is such that } u_n \rightharpoonup u \quad \text{in} \quad X \quad \Longrightarrow \\ \qquad \limsup j^0(u_n; v - u_n) \leq j^0(u; v - u) \quad \forall\, v \in X. \end{array} \right.$$

$$(10) \quad \begin{cases} \pi\colon X \to Y \text{ is such that} \\ v_n \rightharpoonup v \quad \text{in} \quad X \quad \Longrightarrow \quad \pi v_n \to \pi v \quad \text{in} \quad Y. \end{cases}$$

Theorem 2. *Assume that* (1)–(10) *hold and let* $\{f_n\} \subset Y$, $\{g_n\} \subset (0, +\infty)$, $f \in Y$, $g > 0$. *Then,*

$$f_n \rightharpoonup f \quad \text{in} \quad Y, \quad g_n \to g \quad \Longrightarrow \quad u(f_n, g_n) \to u(f, g) \quad \text{in} \quad X.$$

Proof. i) The solution $u = u(f, g)$ of Problem $(\mathcal{P})$ satisfies the bound

$$\|u\|_X \leq \frac{1}{m_A - \alpha_j} \left(\|A0_X\|_{X^*} + d_0 \|f\|_Y + c_0 \right).$$

ii) Let $\{f_n\} \subset Y$, and let $g > 0$. Then,

$$f_n \rightharpoonup f \quad \text{in} \quad Y \quad \Longrightarrow \quad u(f_n, g) \to u(f, g) \quad \text{in} \quad X.$$

iii) Let $\{f_n\} \subset Y$ be a bounded sequence and let $\{g_n\} \subset (0, +\infty)$, $g > 0$. Then,

$$g_n \to g \quad \Longrightarrow \quad u(f_n, g_n) - u(f_n, g) \to 0_X \quad \text{in} \quad X.$$

iv) We apply steps ii) and iii) to conclude the proof. $\square$

IV. TWO OPTIMAL CONTROL PROBLEMS

First Optimal Control Problem

Let $g > 0$ be given and consider the set of admissible pairs

$$\mathcal{V}_{ad}^1 = \{\, (u, f) \in K_g \times Y \text{ such that } (\mathcal{P}) \text{ holds} \,\}.$$

Remark : $(u, f) \in \mathcal{V}_{ad}^1 \Longleftrightarrow f \in Y$ and $u = u(f, g)$.

Problem $\mathcal{Q}_1$. *Find* $(u^*, f^*) \in \mathcal{V}_{ad}^1$ *such that*

$$\mathcal{L}_1(u^*, f^*) = \min_{(u,f)\in\mathcal{V}_{ad}^1} \mathcal{L}_1(u, f).$$

Assume that

$$(11) \qquad \mathcal{L}_1(u,f) = U(u) + F(f) \qquad \forall\, u \in X,\ f \in Y,$$

$$(12) \quad \begin{cases} U : X \to \mathbb{R} \text{ is continuous, bounded and positive, i.e.,} \\ \text{(a)} \quad v_n \to v \quad \text{in} \quad X \quad \Longrightarrow \quad U(v_n) \to U(v). \\ \text{(b)} \quad U \text{ maps bounded sets in } X \text{ into bounded sets in } \mathbb{R}. \\ \text{(c)} \quad U(v) \geq 0 \quad \forall\, v \in X. \end{cases}$$

$$(13) \quad \begin{cases} F : Y \to \mathbb{R} \text{ is l.s.c., positive and coercive, i.e.,} \\ \text{(a)} \quad f_n \rightharpoonup f \quad \text{in} \quad Y \quad \Longrightarrow \quad \liminf F(f_n) \geq F(f). \\ \text{(b)} \quad F(f) \geq 0 \quad \forall\, f \in Y. \\ \text{(c)} \quad \|f_n\|_Y \to +\infty \quad \Longrightarrow \quad F(f_n) \to +\infty. \end{cases}$$

Theorem 3. *Assume that* (1)–(13) *hold and let* $g > 0$ *be given. Then, there exists at least one solution* $(u^*, f^*) \in \mathcal{V}^1_{ad}$ *to Problem* $\mathcal{Q}_1$.

Proof. Let

$$\theta = \inf_{(u,f) \in \mathcal{V}^1_{ad}} \mathcal{L}_1(u, f) \in \mathbb{R} \tag{14}$$

and let $\{(u_n, f_n)\} \subset \mathcal{V}^1_{ad}$ be a minimizing sequence for the functional $\mathcal{L}_1$, i.e.

$$\lim \mathcal{L}_1(u_n, f_n) = \theta. \tag{15}$$

We argue by contradiction and prove that the sequence $\{f_n\}$ is bounded in Y. Therefore there exists $f^* \in Y$ such that, passing to a subsequence still denoted $\{f_n\}$, we have

$$f_n \rightharpoonup f^* \quad \text{in} \quad Y \quad \text{as} \quad n \to +\infty. \tag{16}$$

Let $u^* = u(f^*, g)$. Then, by the definition of the set $\mathcal{V}^1_{ad}$ we have

$$(u^*, f^*) \in \mathcal{V}^1_{ad}. \tag{17}$$

Moreover, using (16) and Theorem 2 it follows that

$$u_n \to u^* \quad \text{in} \quad X \quad \text{as} \quad n \to +\infty. \tag{18}$$

We now use the weakly l.s.c. of $\mathcal{L}_1$ to deduce that

$$\liminf \; \mathcal{L}_1(u_n, f_n) \geq \mathcal{L}_1(u^*, f^*). \tag{19}$$

It follows now from (15) and (19) that $\theta \geq \mathcal{L}_1(u^*, f^*)$. In addition, (14) and (17) yield $\theta \leq \mathcal{L}_1(u^*, f^*)$. We now combine these inequalities to conclude the proof. □

Second Optimal Control Problem

Let $f \in Y$ and $W = [g_0, \infty)$ where $g_0 > 0$ is given. Define the set of admissible pairs by

$$\mathcal{V}_{ad}^2 = \{\, (u, g) \in K_g \times W \text{ such that } (\mathcal{P}) \text{ holds} \,\}.$$

Remark : $(u, g) \in \mathcal{V}_{ad}^2 \iff g \in W$ and $u = u(f, g)$.

Problem $\mathcal{Q}_2$. *Find $(u^*, g^*) \in \mathcal{V}_{ad}^2$ such that*

$$\mathcal{L}_2(u^*, g^*) = \min_{(u,g) \in \mathcal{V}_{ad}^2} \mathcal{L}_2(u, g).$$

Assume that

$$(20)\qquad \mathcal{L}_2(u,f) = U(u) + G(g) \qquad \forall\, u \in X,\ g \in W.$$

$$(21)\quad \begin{cases} G : W \to \mathbb{R} \text{ is l.s.c., positive and coercive, i.e.,} \\ \text{(a)}\quad g_n \rightharpoonup g \quad \Longrightarrow \quad \liminf\ G(g_n) \geq G(g). \\ \text{(b)}\quad G(g) \geq 0 \quad \forall\, g \in W. \\ \text{(c)}\quad g_n \to +\infty \quad \Longrightarrow \quad G(g_n) \to +\infty. \end{cases}$$

Theorem 4. *Assume that* (1)–(10), (12), (20), (21) *hold and let* $f \in Y$. *Then, there exists at least one solution* $(u^*, g^*) \in \mathcal{V}^2_{ad}$ *to Problem* $\mathcal{Q}_2$.

The proof of Theorem 4 is based on arguments similar to those used on the proof of Theorem 3.

Convergence results for the optimal pairs

We focus on the dependence of the optimal pairs of problems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect the data g and f, respectively.

We start with the study of **Problem $\mathcal{Q}_1$** and, to this end, we work on the hypothesis of Theorem 3. Let g_n be a perturbation of g, denote $K_n = g_n K$ and consider the following perturbation of **Problem $\mathcal{P}$**.

Problem $\mathcal{P}_n$. *Given $f \in Y$ and $g_n > 0$, find $u_n \in K_{g_n}$ such that*

$$\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n)$$

$$\geq (f, \pi v - \pi u_n)_Y \quad \forall\, v \in K_{g_n}.$$

It follows from Theorem 1 that for each $f \in Y$ and $g_n > 0$ there exists a unique solution $u_n = u(f, g_n)$ to **Problem** $\mathcal{P}_n$. We define set of admissible pairs by

$$\mathcal{V}_{ad}^{1n} = \{\, (u_n, f) \in K_{g_n} \times Y \text{ such that } (\mathcal{P}_n) \text{ holds} \,\}.$$

Then, optimal control problem associated to **Problem** $\mathcal{P}_n$ the following.

Problem $\mathcal{Q}_n^1$**.** *Find* $(u_n^*, f_n^*) \in \mathcal{V}_{ad}^{1n}$ *such that*

$$\mathcal{L}_1(u_n^*, f_n^*) = \min_{(u_n, f_n) \in \mathcal{V}_{ad}^{1n}} \mathcal{L}_1(u_n, f_n). \tag{0.1}$$

Using Theorem 3 it follows that for each $n \in \mathbb{N}$ there exists at least one solution $(u_n^*, f_n^*) \in \mathcal{V}_{ad}^{1n}$ to **Problem** $\mathcal{Q}_n^1$.

Theorem 5. *Let* $\{(u_n^*, f_n^*)\}$ *be a sequence of solutions to* **Problems** $\mathcal{Q}_n^1$ *and assume that* $g_n \to g$. *Then, there exists a subsequence of the sequence* $\{(u_n^*, f_n^*)\}$, *again denoted* $\{(u_n^*, f_n^*)\}$, *and a solution* (u^*, f^*) *to* **Problem** $\mathcal{Q}_1$ *such that*

$$u_n \to u^* \quad \text{in} \quad X \quad \text{and} \quad f_n^* \rightharpoonup f^* \quad \text{in} \quad Y.$$

Proof. We use arguments of coercivity, compactness, and lower semicontinuity.

Remark. A similar convergence result can be obtained in the study of the optimal control **Problem** $\mathcal{Q}_2$.

V. BACK TO THE ONE-DIMENSIONAL EXAMPLE

We consider **Problem** $\mathcal{P}^{1d}$ in the particular case $L = 1$, $\mathcal{F}\varepsilon = E\varepsilon$ with $E > 0$, $p \equiv 0$, $f \in \mathbb{R}$. Also, below we use notation $P = F$.

Note that, since $p \equiv 0$, the weak formulation of this problem is in a form of a **variational inequality.**

Problem $\mathcal{P}^{1d}$. *Find a displacement field* $u\colon [0,1] \to \mathbb{R}$ *and a stress field* $\sigma\colon [0,1] \to \mathbb{R}$ *such that*

$$\sigma(x) = E\,u'(x) \quad \text{for } x \in (0,1),$$

$$\sigma'(x) + f = 0 \quad \text{for } x \in (0,1),$$

$$u(0) = 0,$$

$$u(1) \le g, \qquad \left.\begin{array}{ll} \sigma(1) = 0 & \text{if } \; u(1) < 0 \\ -F < \sigma(1) < 0 & \text{if } \; u(1) = 0 \\ \sigma(1) = -F & \text{if } \; 0 < u(1) < g \\ \sigma(1) \le -F & \text{if } \; u(1) = g \end{array}\right\}.$$

For the variational analysis of **Problem** $\mathcal{P}^{1d}$ we use the space

$$V = \{\, v \in H^1(0,1) \ : \ v(0) = 0 \,\}$$

and the set of admissible displacement field defined by

$$K_g = \{\, u \in V \mid u(1) \leq g \,\}.$$

The variational formulation of **Problem** $\mathcal{P}^{1d}$ is the following.

Problem $\mathcal{P}^{1d}_V$. *Find a displacement field $u \in K_g$ such that*

$$\int_0^1 Eu'(v'-u')\,dx + Fv(1)^+ - Fu(1)^+ \geq \int_0^1 f\,(v-u)\,dx \quad \forall\, v \in K_g.$$

A simple calculation allows to solve **Problems $\mathcal{P}^{1d}$ and $\mathcal{P}_V^{1d}$**. Four cases are possible, described below, together with the corresponding mechanical interpretations.

a) **The case $f < 0$.** In this case the body force acts into the oposite direction of the foundation and the solution of Problem $\mathcal{P}^{1d}$ is given by

$$\begin{cases} \sigma(x) = -fx + f, \\ u(x) = -\frac{f}{2E}\,x^2 + \frac{f}{E}\,x \end{cases} \qquad \forall\, x \in [0,1].$$

We have $u(1) < 0$ and $\sigma(1) = 0$ which shows that there is separation between the rod and the foundation and, therefore, there is no reaction on the point $x = 1$. This case corresponds to Figure 3 a).

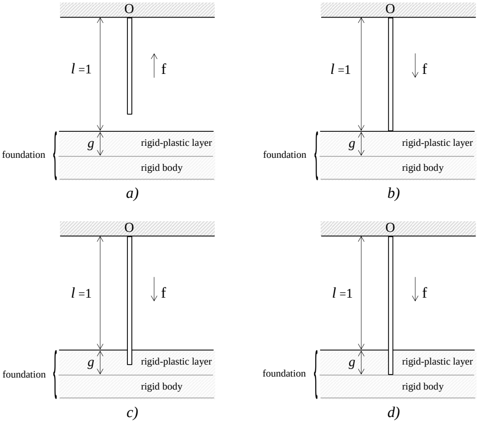

Figure: 3. The rod in contact with a foundation:

a) The case $f < 0$; b) The case $0 \leq f < 2F$;

c) The case $2F \leq f < 2Eg + 2F$; d) The case $2Eg + 2F \leq f$.

b) **The case $0 \leq f < 2F$.** In this case the body force pushses the rod towards the foundation and the solution of Problem $\mathcal{P}^{1d}$ is given by

$$\begin{cases} \sigma(x) = -fx + \frac{f}{2}, \\ u(x) = -\frac{f}{2E}\, x^2 + \frac{f}{2}\, x \end{cases} \qquad \forall\, x \in [0,1].$$

We have $u(1) = 0$ and $-F < \sigma(1) \leq 0$ which shows that the rod is in contact with the foundation and the reaction of the foundation is towards the rod. Nevertheless, there is no penetration, since the magnitude of the stress in $x = 1$ is under the yield limit F and, therefore, the rigid-plastic layer behaves like a rigid. This case corresponds to Figure 3 b).

c) **The case $2F \leq f < 2Eg + 2F$.** In this case the solution of Problem $\mathcal{P}^{1d}$ is given by

$$\begin{cases} \sigma(x) = -fx + f - F, \\ u(x) = -\frac{f}{2E}\, x^2 + \frac{f-F}{E}\, x \end{cases} \qquad \forall\, x \in [0,1].$$

We have $0 \leq u(1) < g$ and $-\sigma(1) = F$. This shows that the magnitude of the stress in $x = 1$ reached the yield limit and, therefore, there is penetration into the rigid-plastic layer which now behaves plastically. Nevertheless, the penetration is partially, since $u(1) < g$. This case corresponds to Figure 3 c).

d) **The case $2Eg + 2F \leq f$.** In this case the solution of Problem $\mathcal{P}^{1d}$ is given by

$$\begin{cases} \sigma(x) = -fx + \frac{2Eg+f}{2}, \\ u(x) = -\frac{f}{2E}\, x^2 + \frac{2Eg+f}{2E}\, x \end{cases} \qquad \forall\, x \in [0,1].$$

We have $0 \leq u(1) = g$ and $\sigma(1) \leq -F$ which shows that the rigid-plastic layer is completely penetrated and the point $x = 1$ reaches the rigid body. The magnitude of the reaction in this point is larger then the yield limit F, i.e. $|\sigma(1)| \geq F$ since, besides the reaction of the rigid-plastic layer, we add the reaction of the rigid body which now becomes active. This case corresponds to Figure 3 d).

We now formulate the **optimal control problem** $\mathcal{Q}_2$ in the one-dimensional case of **Problem** $\mathcal{P}^{1d}$, with the cost functional

$$\mathcal{L}_2(u, g) = \alpha\, |u(1) - \phi| + \beta\, |g|,$$

where $\phi \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $U = [g_0, \infty)$ with $g_0 > 0$.

Problem $\mathcal{Q}_2^{1d}$. *Find* $(u^*, g^*) \in \mathcal{V}_{ad}^2$ *such that*

$$\mathcal{L}_2(u^*, g^*) = \min_{(u,g) \in \mathcal{V}_{ad}^2} \mathcal{L}_2(u, g).$$

Mechanical interpretation : given f, we are looking for a thickness $g \in U$ such that the displacement of the rod in $x = 1$ is as close as possible to the "desired displacement" ϕ. Furthermore, this choice has to fulfill a minimum expenditure condition.

We now take $E = 1$, $f = 10$, $F = 2$, $\phi = 4$ and $g_0 = 1$ which implies that $U = [1, +\infty)$. It is easy to see that

$$u(x) = \begin{cases} -5x^2 + (g+5)\,x & \text{if } \; 1 \leq g \leq 3, \\ -5x^2 + 8x & \text{if } \; g > 3 \end{cases} \tag{22}$$

for all $x \in [0,1]$ and, therefore,

$$\mathcal{L}_2(u, g) = \begin{cases} (\beta - \alpha)\,g + 4\alpha & \text{if } \; 1 \leq g \leq 3, \\ \beta\,g + \alpha & \text{if } \; g > 3. \end{cases}$$

Conclusions

a) If $\beta > \alpha > 0$ then the optimal control problem $\mathcal{Q}_2^{1d}$ has a unique solution (u^*, g^*) where $g^* = 1$ and u^* is given by (22) with $g = g^*$.

b) If $\beta = \alpha$ then the optimal control problem $\mathcal{Q}_2^{1d}$ has an infinity of solutions of the form (u^*, g^*) where g^* is any value in the interval $[1, 3]$ and u^* is given by (22) with $g = g^*$.

c) If $0 < \beta < \alpha$ then the optimal control problem $\mathcal{Q}_2^{1d}$ has a unique solution (u^*, g^*) where $g^* = 3$ and u^* is given by (22) with $g = g^*$.

VI. d-DIMENSIONAL EXAMPLE ($d = 2, 3$)

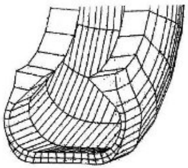

Fig. 5.20 – Écrasement statique - $F = 6$ tonnes

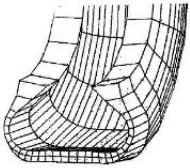

Fig. 5.21 – Écrasement statique - $F = 10$ tonnes

Fig. 5.23 – Écrasement statique - $F = 18$ tonnes

Figure: 4. Tire of the plane at landing.

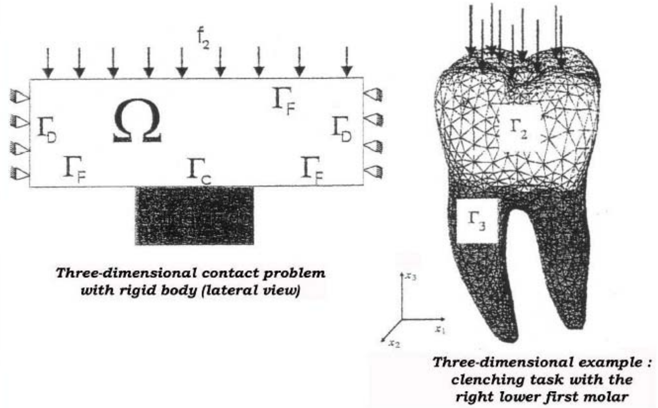

Figure: 5. Some examples of contact problems.

Problem $\mathcal{P}$. *Find the displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and the stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$
\begin{aligned}
\boldsymbol{\sigma} &= \mathcal{F}\varepsilon(\boldsymbol{u}) && \text{in } \Omega, \\
\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 &= \boldsymbol{0} && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_1, \\
\boldsymbol{\sigma\nu} &= \boldsymbol{f}_2 && \text{on } \Gamma_2, \\
&\left.\begin{aligned} & u_\nu \le g, \quad \sigma_\nu + k_\nu(u_\nu) \le 0, \\ & (\sigma_\nu + k_\nu(u_\nu))(u_\nu - g) = 0. \end{aligned}\right\} && \text{on } \Gamma_3, \\
\boldsymbol{\sigma}_\tau &= \boldsymbol{0} && \text{on } \Gamma_3.
\end{aligned}
$$

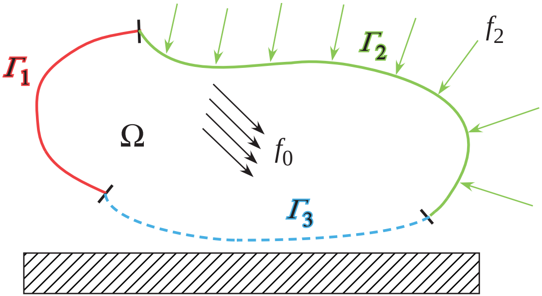

Figure: 6. Physical setting.

Notation

Ω - bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$;

Γ - boundary of Ω ;

Γ_1, Γ_2, Γ_3 - partition of Γ such that *meas* $\Gamma_1 > 0$;

$\boldsymbol{\nu}$ - unit outward normal on Γ ;

$\mathbb{S}^d$ - space of second order symmetric tensors on $\mathbb{R}^d$;

$\boldsymbol{\varepsilon}$ - the deformation operator;

v_ν, $\boldsymbol{v}_\tau$ - *normal* and *tangential* components of $\boldsymbol{v}$ on Γ;

σ_ν, $\boldsymbol{\sigma}_\tau$ - *normal* and *tangential* components of $\boldsymbol{\sigma}$ on Γ;

$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : v_i = 0 \text{ on } \Gamma_1 \,\},$

Inner product:

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \varepsilon(\boldsymbol{u}) \cdot \varepsilon(\boldsymbol{v})\, dx,$$

V^*- dual of the space V,

$\langle \cdot, \cdot \rangle_{V^* \times V}$ - duality pairing.

Here

$$k_\nu(r) = q_\nu(r) + p_\nu(r) \qquad \text{for all} \quad r \in \mathbb{R},$$

where $q_\nu : \mathbb{R} \to \mathbb{R}$ is a monotone function and $p_\nu : \mathbb{R} \to \mathbb{R}$ is a nonmonotone locally Lipschitz function. Define $\varphi_\nu : \mathbb{R} \to \mathbb{R}$, $j_\nu : \mathbb{R} \to \mathbb{R}$ and $U \subset V$ by equalities

$$\varphi_\nu(r) = \int_0^r q_\nu(s)\, ds, \quad j_\nu(r) = \int_0^r p_\nu(s)\, ds \quad \forall\, r \in \mathbb{R},$$

$$U = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}.$$

Variational formulation: *Find $\boldsymbol{u} \in U$ such that*

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) + j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \forall\, \boldsymbol{v} \in U$$

where

$$A\colon V \to V^*, \quad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$\varphi\colon V \to \mathbb{R}, \quad \varphi(\boldsymbol{v}) = \int_{\Gamma_3} \varphi_\nu(v_\nu)\, d\Gamma \quad \text{for } \boldsymbol{v} \in V,$$

$$j\colon V \to \mathbb{R}, \quad j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(\boldsymbol{v}_\nu)\, d\Gamma \quad \text{for } \boldsymbol{v} \in V,$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_3} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, dx \quad \text{for } \boldsymbol{v} \in V.$$

Remark. Our abstract results (existence, uniqueness, convergence, control) can be applied in the study of this problem, under appropriate assumption on the data.

Numerical simulations

$$q_\nu(r) = \alpha\beta r_+, \quad r \in \mathbb{R},$$

$$p_\nu(r) = \alpha\, p(r) \quad \text{where} \quad p(r) = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } r \in [0,1], \\ 2 - r & \text{if } r \in (1,2], \\ r - 2 & \text{if } r > 2. \end{cases}$$

If $0 \le \beta < 1$ then $k_\nu = q_\nu + p_\nu$ is not a monotone function $\Longrightarrow$ **purely hemivariational inequality**.

If $\beta \ge 1$ then $k_\nu = q_\nu + p_\nu$ is a monotone function $\Longrightarrow$ **purely variational inequality**.

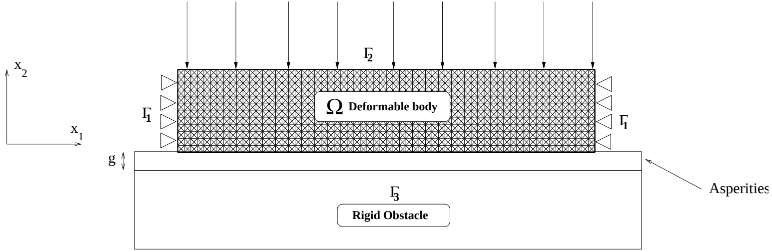

Figure: 7. Reference configuration of the two-dimensional body.

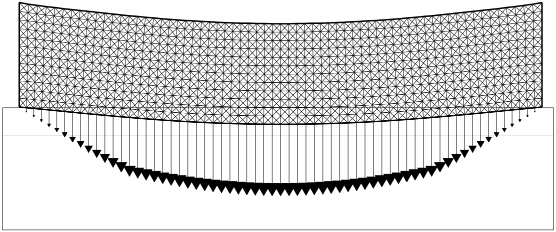

Figure: 8. Deformed mesh and interface forces for $\alpha = 80$ and $\beta = 2$ (normal compliance, monotone case).

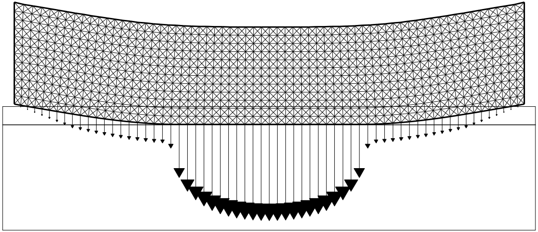

Figure: 9. Deformed mesh and interface forces for $\alpha = 20$ and $\beta = 2$ (unilateral contact, monotone case).

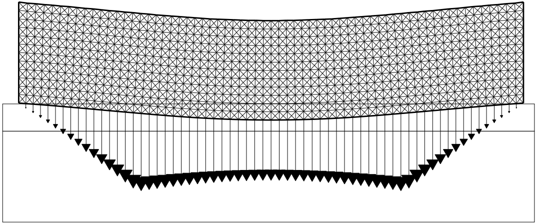

Figure: 10. Deformed mesh and interface forces for $\alpha = 70$ and $\beta = 0.5$ (normal compliance, nonmonotone case).

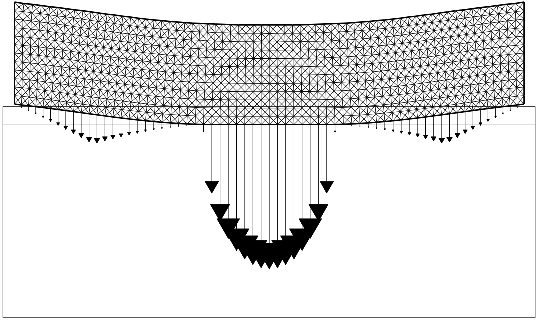

Figure: 10. Deformed mesh and interface forces for $\alpha = 50$ and $\beta = 0.5$ (unilateral contact, nonmonotone case).

VII. REFERENCES

- **M. Sofonea**, *Optimal Control of a Class of Variational-Hemivariational Inequalities in Reflexive Banach Spaces*, submitted.
- W. Han, **M. Sofonea**, & M Barboteu, Numerical Analysis of Elliptic Hemivariational Inequalities, *SIAM Journal of Numerical Analysis* 55 (2017), 640–663.
- S. Migórski, A. Ochal & **M. Sofonea**, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.
- **M. Sofonea** & S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Taylor & Francis, 2017, to appear.

Thank you for your attention!