Exit time for Self-Interacting Diffusions

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Outline

$$\begin{cases} X_t^\sigma = X_0^\sigma - \int_0^t \nabla V(X_s^\sigma)\,\mathrm{d}s - \int_0^t \nabla F * \mu_s^\sigma(X_s^\sigma)\,\mathrm{d}s + \sigma W_t, \\ \mu_t^\sigma = \frac{1}{t}\int_0^t \delta_{X_s^\sigma}\,\mathrm{d}s\,, \\ X_0^\sigma = x_0 \in \mathbb{R}^d \ \text{ a.s.}; \end{cases}$$

1. $V : \mathbb{R}^d \to \mathbb{R}$ – **confinement potential**,
2. $F : \mathbb{R}^d \to \mathbb{R}$ – **interaction potential**,
3. $\mu_t^\sigma = \mu_t^\sigma(\omega)$ – **occupation measure**,
4. W – d-dimensional Brownian motion
5. δ – Dirac measure.

Self-Interacting Diffusion

Alternative Formulations

1. Without the occupation measure:

$$\begin{cases} \mathrm{d}X_t^\sigma = -\nabla V(X_t^\sigma)\,\mathrm{d}t - \frac{1}{t}\left(\int_0^t \nabla F(X_t^\sigma - X_s^\sigma)\,\mathrm{d}s\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \\ X_0^\sigma = x_0 \in \mathbb{R}^d \ \text{ a.s.} \end{cases}$$

2. Starting at some $\mathbf{x} = (x_0, \mu_0, t_0)$

$$\begin{cases} \mathrm{d}X_t^{\sigma,\mathbf{x}} = -\nabla V(X_t^{\sigma,\mathbf{x}})\,\mathrm{d}t - \nabla F * \mu_t^\sigma(X_t^{\sigma,\mathbf{x}})\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \\ \mu_t^{\sigma,\mathbf{x}} = \frac{t_0}{t_0+t}\mu_0 + \frac{1}{t_0+t}\int_0^t \delta_{X_s^{\sigma,\mathbf{x}}}\,\mathrm{d}s\,, \\ X_0^\sigma = x_0 \in \mathbb{R}^d \text{ a.s..} \end{cases} \tag{1}$$

Self-Interacting Diffusion

A bit of History

1. "Brownian polymers": [DR92], [BLR02]
2. Self-Interacting Diffusion: [BLR02], [BR05], [KK10]

Theorem ([KK10])

Under the following assumptions:

1. $V \in \mathcal{C}^2(\mathbb{R}^d)$, $F \in \mathcal{C}^2(\mathbb{R}^d)$ *and* $V \geq 0, \ F \geq 0$,
2. V *and* F *have at most a polynomial growth,*
3. F *is rotationaly invariant, i.e.* $F(x) = \phi(|x|)$
4. V *and* F *are uniformly strictly convex*

Then there exists a unique invariant probability density $\rho_\infty : \mathbb{R}^d \to \mathbb{R}$, *such that almost surely*

$$\mu_t \xrightarrow[t\to\infty]{*-weakly} \rho_\infty(x)\,\mathrm{d}x$$

Outline

Simulations

1. Deterministic system
2. Convex/Convex case
3. Convex/Concave case

Outline

Exit Problem

Formulation

Given open domain $G \subset \mathbb{R}^d$, we want to estimate the probability of exiting this domain, i.e. the following stopping time

$$\tau_G^\sigma := \inf\{t > 0 : X_t^\sigma \in \partial G\}.$$

Usual assumptions on G:

- There is a unique point of minimum of V inside G
- G is bounded
- Stability of G

Exit Problem
Existing results

1. Itô diffusion: [FW98], [DZ10]

Exit Problem

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$$Z_t^\sigma = x_0 - \int_0^t \nabla V(Z_s^\sigma)\,\mathrm{d}s + \sigma W_t$$

Exit Problem
Existing results

1. Itô diffusion: [FW98], [DZ10]

$$Z_t^\sigma = x_0 - \int_0^t \nabla V(Z_s^\sigma)\, \mathrm{d}s + \sigma W_t$$

Assumptions:

- $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $V \geq 0$,
- G – bounded domain, that is stable under V and a is a unique stable point of attraction inside G

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Assumptions:

- $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $V \geq 0$,
- G – bounded domain, that is stable under V and a is a unique stable point of attraction inside G

Theorem

Under the assumptions above,

$$\lim_{\sigma \to 0} \mathbb{P}\left(\exp\left\{\frac{2}{\sigma^2}(H-\delta)\right\} \leq \tau_G^\sigma \leq \exp\left\{\frac{2}{\sigma^2}(H+\delta)\right\}\right) = 1\,,$$

where $H := \inf_{x \in \partial G}\{V(x) - V(a)\}$.

Exit Problem

Existing results

1. Itô diffusion: [FW98], [DZ10]
2. McKean-Vlasov diffusion: [HIP08], [Tug16]

Motivation

Computer Science

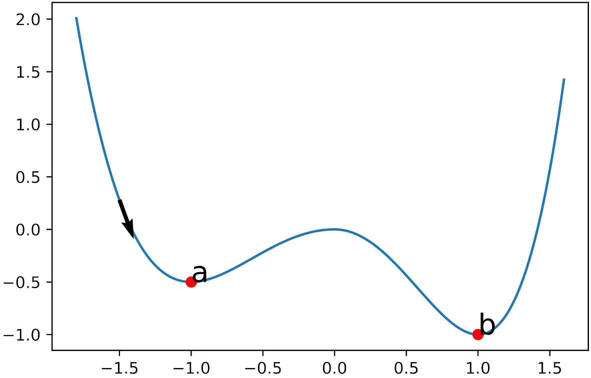

Figure: Double-well Landscape

Outline

1. $V \in \mathcal{C}^2(\mathbb{R}^d)$, $F \in \mathcal{C}^2(\mathbb{R}^d)$ and $V \geq 0$, $F \geq 0$,
2. V and F have at most a polynomial growth
3. F is rotationaly invariant, i.e. $F(x) = \phi(|x|)$
4. V is uniformly strictly convex and its global minimum is at a
5. F is uniformly strictly convex and its global minimum is at 0
6. Open domain G is stable by $-\nabla V - \nabla F(\cdot - a)$ and the orbit of $\psi_t = x_0 - \int_0^t \nabla V(\psi_s)\,\mathrm{d}s - \int_0^t \int_0^s \nabla F(\psi_s - \psi_u)\,\mathrm{d}u\,\mathrm{d}s$ stays inside the domain

Theorem

Under the Assumptions above,

$$\lim_{\sigma\to 0}\mathbb{P}\left(\exp\left\{\frac{2}{\sigma^2}\left(H_{SI}-\delta\right)\right\}\leq\tau_G^\sigma\leq\exp\left\{\frac{2}{\sigma^2}\left(H_{SI}+\delta\right)\right\}\right)=1\,,$$

where $H_{SI} := \inf_{x\in\partial G}\left\{V(x)+F(x-a)-V(a)\right\} > H.$

Outline

General Case
Revision of Assumptions

1. $V \in \mathcal{C}^2(\mathbb{R}^d)$, $F \in \mathcal{C}^2(\mathbb{R}^d)$ and $V \geq 0$, $F \geq 0$,
2. V and F have at most a polynomial growth,
3. The unique minimum of V in G is at a,
4. ∇F is equal to 0 at 0,
5. Open domain G is stable by $-\nabla V - \nabla F(\cdot - a)$ and the orbit of $\psi_t = x_0 - \int_0^t \nabla V(\psi_s)\,\mathrm{d}s - \int_0^t \int_0^s \nabla F(\psi_s - \psi_u)\,\mathrm{d}u\,\mathrm{d}s$ stays inside the domain

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Thank you for your attention!

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