Propagation of chaos in a network of Fitz-Hugh-Nagumo

Laetitia Colombani,
Joint work with Pierre Le Bris

IMT

19th May 2022,
Workshop Metastability, mean-field particle systems and non linear processes

1. Model: FitzHugh-Nagumo

2. Propagation of chaos and our results

3. Strategy

What do we model?

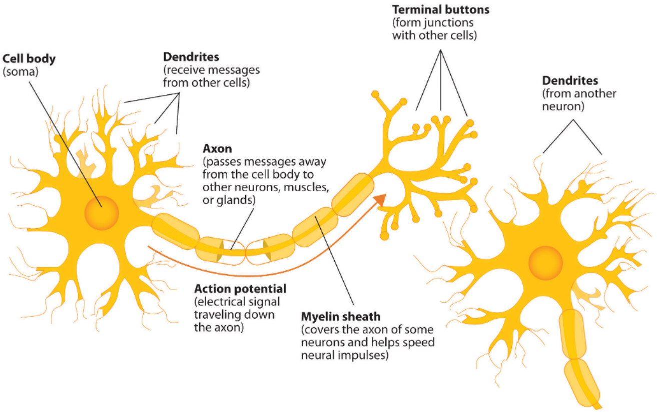

Figure: Neurons and a connexion

What do we model?

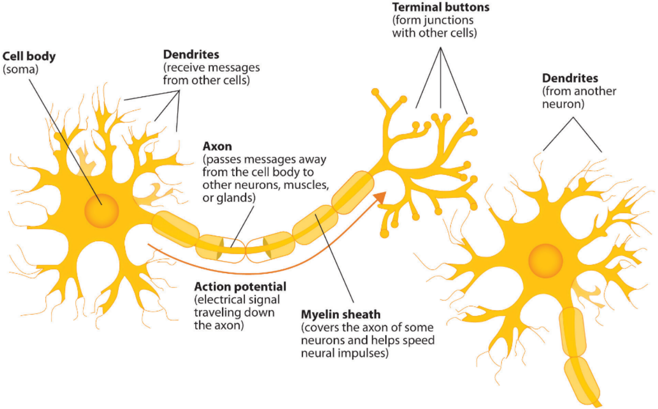

Figure: Neurons and a connexion

Main elements:

- Membrane potential
- Incoming signal
- Outgoing signal

What do we model?

- Model membrane potential
- Noise from the presynaptic current
- Noise from the conductance dynamic
- Transmission of potential between neurons via synapses

What do we model?

- Model membrane potential X
- Noise from the presynaptic current
- Noise from the conductance dynamic
- Transmission of potential between neurons via synapses

Deterministic model

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt \\ dC_t = (\gamma X_t - C_t + \beta)dt \\ Z_t = (X_t, C_t). \end{cases}$$

C is a recovery variable. α, β, γ fixed.

What do we model?

- Model membrane potential X
- Noise from the presynaptic current B^X
- Noise from the conductance dynamic B^C
- Transmission of potential between neurons via synapses

Stochastic model

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt + \sigma_x dB_t^X \\ dC_t = (\gamma X_t - C_t + \beta)dt + \sigma_c dB_t^C \\ Z_t = (X_t, C_t). \end{cases}$$

C is a recovery variable.

What do we model?

- Model membrane potential X
- Noise from the presynaptic current B^X
- Noise from the conductance dynamic B^C
- Transmission of potential between neurons via synapses: interaction K_X and K_C

Stochastic model with interactions

We consider N neurons and for each $i \leq N$, $X^{i,N}$ and $C^{i,N}$ their quantities.

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)dt + \sigma_x dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j)dt + \sigma_c dB_t^{i,C} \\ Z_t^{i,N} = (X_t^{i,N}, C_t^{i,N}). \end{cases} \quad (1)$$

State of the art : FitzHugh-Nagumo

- León, Samson (2018). *"Hypoelliptic stochastic FitzHugh–Nagumo neuronal model: mixing, up-crossing and estimation of the spike rate"*
  - Noise only on C (conductance dynamic)
  - Study of equations for one neuron.

State of the art : FitzHugh-Nagumo

- León, Samson (2018). *"Hypoelliptic stochastic FitzHugh–Nagumo neuronal model: mixing, up-crossing and estimation of the spike rate"*
  - Noise only on C (conductance dynamic)
  - Study of equations for one neuron.
- Mischler, Quininao, Touboul (2016). *"On a kinetic FitzHugh-Nagumo model of neuronal network"*.
  - Noise only on X (presynaptic current).
  - Linear interaction on X ($K_C = 0$ and $K_X(x, c) = \lambda x$).
  - Study on N neurons (propagation of chaos, existence, uniqueness)

State of the art : FitzHugh-Nagumo

- León, Samson (2018). *"Hypoelliptic stochastic FitzHugh–Nagumo neuronal model: mixing, up-crossing and estimation of the spike rate"*
  - Noise only on C (conductance dynamic)
  - Study of equations for one neuron.
- Mischler, Quininao, Touboul (2016). *"On a kinetic FitzHugh-Nagumo model of neuronal network"*.
  - Noise only on X (presynaptic current).
  - Linear interaction on X ($K_C = 0$ and $K_X(x, c) = \lambda x$).
  - Study on N neurons (propagation of chaos, existence, uniqueness)
- Baladron, Fasoli, Faugeras, Touboul (2012). *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons*
  - Noise only on X
  - Interaction more complicated, on X
  - Propagation of chaos and convergence

State of the art : FitzHugh-Nagumo

In this work: $\sigma_C > 0$ or/and $\sigma_X > 0$.
Lipschitz interaction.

1. Model: FitzHugh-Nagumo
2. Propagation of chaos and our results
3. Strategy

Idea of propagation of chaos

We consider

- $N \to +\infty$
- Fixed $k \in \mathbb{N}^*$
- t may be large

What is the behavior of k particles following the SDE when $N \mapsto +\infty$?

Intuition: if particles are independent at $t = 0$, the chaos generated by all N particles (very large) makes the k first neurons stay "independent".

Idea of propagation of chaos

We consider

- $N \to +\infty$
- Fixed $k \in \mathbb{N}^*$
- t may be large

We fix initial distribution $(\mu_0)^{\otimes N}$.
We denote $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons in a network of N neurons with this initial distribution.
We measure the distance between $\mu_t^{k,N}$ and $\bar{\mu}^{\otimes k}$ where $\bar{\mu}_t$ is a specific measure.

Definition of $\bar{\mu}_t$?

We denote $K * \nu(x) = \int K(x-y)\nu(dy)$.
$\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j)$ can be seen as $K_X * \left(\frac{1}{N}\sum_{j=1}^{N}\delta_{Z_t^j}\right)$ applied on Z_t^i.

Definition of $\bar{\mu}_t$?

We denote $K * \nu(x) = \int K(x-y)\nu(dy)$.
$\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)$ can be seen as $K_X * \left(\frac{1}{N}\sum_{j=1}^N \delta_{Z_t^j}\right)$ applied on Z_t^i.
We construct a process for *one particule* (without interaction).

Definition of $\bar{\mu}_t$?

We denote $K * \nu(x) = \int K(x-y)\nu(dy)$.
$\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j)$ can be seen as $K_X * \left(\frac{1}{N}\sum_{j=1}^{N}\delta_{Z_t^j}\right)$ applied on Z_t^i.
We construct a process for *one particule* (without interaction).

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j)dt + \sigma_x dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^{N} K_C(Z_t^i - Z_t^j)dt + \sigma_c dB_t^{i,C} \\ Z_t^{i,N} = (X_t^{i,N}, C_t^{i,N}). \end{cases}$$

$$\Downarrow$$

$$\begin{cases} d\bar{X}_t = (\bar{X}_t - (\bar{X}_t)^3 - \bar{C}_t - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_x d\bar{B}_t^X \\ d\bar{C}_t = (\gamma \bar{X}_t - \bar{C}_t + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_c d\bar{B}_t^C \\ \bar{Z}_t = (\bar{X}_t, \bar{C}_t) \\ \bar{\mu}_t = \mathrm{Law}((\bar{X}_t, \bar{C}_t)), \end{cases}$$

Non-linear stochastic differential equation of *McKean-Vlasov* type

Definition of $\bar{\mu}_t$?

We denote $K * \nu(x) = \int K(x-y)\nu(dy)$.
$\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)$ can be seen as $K_X * \left(\frac{1}{N}\sum_{j=1}^N \delta_{Z_t^j}\right)$ applied on Z_t^i.

For $1 \leq i \leq N$,

$$\begin{cases} d\bar{X}_t^i = (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_x d\bar{B}_t^{i,X} \\ d\bar{C}_t^i = (\gamma \bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_c d\bar{B}_t^{i,C} \\ \bar{Z}_t^i = (\bar{X}_t^i, \bar{C}_t^i) \\ \bar{\mu}_t = \text{Law}((\bar{X}_t^1, \bar{C}_t^1)), \end{cases}$$

Then $\bar{\mu}_t^{\otimes k}$ is the law of $(\bar{Z}_t^1, \dots, \bar{Z}_t^k)$.

- Same type of drift.
- Same idea for the interaction.
- Different noise.

Result: Propagation of chaos

Let K_X be L_X-Lipschitz, K_C be L_C-Lipschitz. Let's notice σ_X or σ_C can be null.

Theorem (Non uniform in time propagation of chaos)

There exist explicit $C_1, C_2 > 0$, such that for all good probability measures μ_0 on $\mathbb{R}^2$

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq C_1 e^{C_2 t} \frac{k}{\sqrt{N}}.$$

Result: Propagation of chaos

Let K_X be L_X-Lipschitz, K_C be L_C-Lipschitz. Let's notice σ_X or σ_C can be null.

Theorem (Non uniform in time propagation of chaos)

There exist explicit $C_1, C_2 > 0$, such that for all good probability measures μ_0 on $\mathbb{R}^2$

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq C_1 e^{C_2 t} \frac{k}{\sqrt{N}}.$$

Theorem (Uniform in time propagation of chaos)

Under some condition on L_X and L_C, there exist explicit $B_1 > 0$, such that for all good probability measures μ_0 on $\mathbb{R}^2$

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_1 \frac{k}{\sqrt{N}}.$$

Strategy: Coupling method

Wasserstein distance

$$\mathcal{W}_1(\nu, \nu') = \inf_{\pi \in \Pi(\nu, \nu')} \int d(z, z') \pi(dz, dz')$$

Strategy: Coupling method

Wasserstein distance

$$\mathcal{W}_1(\nu, \nu') = \inf_{\pi \in \Pi(\nu, \nu')} \int d(z, z')\pi(dz, dz')$$

Two choices:

- choice of coupling π
- choice of distance (or major bound of) d

Strategy: Coupling method

Naive idea: Synchronous coupling

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|\right]$$

Strategy: Coupling method

Naive idea: Synchronous coupling

$$\mathcal{W}_1\left(\mu_t^{k,N},\bar{\mu}_t^{\otimes k}\right)\leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|X_t^{i,N}-\bar{X}_t^i|+|C_t^{i,N}-\bar{C}_t^i|\right]$$

Study the dynamics $d(X_t^{i,N}-\bar{X}_t^i)$ and $d(C_t^{i,N}-\bar{C}_t^i)$.

Strategy: Coupling method

Naive idea: Synchronous coupling

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|\right]$$

Study the dynamics $d(X_t^{i,N} - \bar{X}_t^i)$ and $d(C_t^{i,N} - \bar{C}_t^i)$.
Define $\bar{B}_t^{i,X} = B_t^{i,X}$ and $\bar{B}_t^{i,C} = B_t^{i,C}$.

Strategy: Coupling method

Naive idea: Synchronous coupling

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|\right]$$

Study the dynamics $d(X_t^{i,N} - \bar{X}_t^i)$ and $d(C_t^{i,N} - \bar{C}_t^i)$.
Define $\bar{B}_t^{i,X} = B_t^{i,X}$ and $\bar{B}_t^{i,C} = B_t^{i,C}$.

$$d(X_t^{i,N} - \bar{X}_t^i) = (\text{Drift on } X_t^{i,N})dt + \frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j)dt - (\text{Drift on } \bar{X}_t^i)dt - K_X * \bar{\mu}_t(\bar{Z}_t^i)dt$$

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

1) Denote $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|$. Then

$$dr_t^i \leq \left[\mathcal{C} r_t^i + \left| \frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| + \left| \frac{1}{N} \sum_{j=1}^{N} K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right] dt.$$

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

1) Denote $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|$. Then

$$dr_t^i \leq \left[\mathcal{C} r_t^i + \left| \frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| + \left| \frac{1}{N} \sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right] dt.$$

2) Decompose interaction part

$$\left| \frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \leq \left| \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| + \left| \frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) \right|.$$

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

2) Decompose interaction part

$$\left|\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \leq \left|\frac{1}{N}\sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| + \left|\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - \frac{1}{N}\sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right|.$$

First sum: bounded by $\sum r_t^j$ thanks to Lipschitz property.

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

2) Decompose interaction part

$$\left|\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \leq \left|\frac{1}{N}\sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| + \left|\frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j) - \frac{1}{N}\sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right|.$$

First sum: bounded by $\sum r_t^j$ thanks to Lipschitz property.

Second sum: we bound expectation of the sum with sort of law of large numbers.

Strategy: Coupling method

Naive idea: Synchronous coupling

Steps:

3) Finally

$$d\mathbb{E}(r_t^i) \leq \left(\mathcal{C}\mathbb{E}(r_t^i) + \mathcal{C}\frac{1}{\sqrt{N}} \left(\mathbb{E}(\|\bar{Z}_t\|_1^2) \right)^{1/2} \right) dt.$$

Good bounds on $\mathbb{E}(\|\bar{Z}_t\|_1^2)$ + Gronwall's lemma $\Rightarrow$ first theorem proved (non uniform in time)

Strategy: Coupling method

Improved idea: Mixed coupling

(Here $\sigma_X > 0$). Consider, for an adequate δ:

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} f(|X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|)\right]$$

Strategy: Coupling method

Improved idea: Mixed coupling

(Here $\sigma_X > 0$). Consider, for an adequate δ:

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} f(|X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|)\right]$$

Study the dynamic $d(X_t^{i,N} - \bar{X}_t^i)$ and $d(C_t^{i,N} - \bar{C}_t^i)$.

Strategy: Coupling method

Improved idea: Mixed coupling

(Here $\sigma_X > 0$). Consider, for an adequate δ:

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} f(|X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|)\right]$$

Study the dynamic $d(X_t^{i,N} - \bar{X}_t^i)$ and $d(C_t^{i,N} - \bar{C}_t^i)$.
In the subspace $\{(X_t^{i,N} - \bar{X}_t^i) = 0\}$: deterministic contraction. $\Rightarrow$ Synchronous coupling
In the orthogonal subspace $\Rightarrow$ Reflection coupling (maximization of variance)

Strategy: Coupling method

Improved idea: Mixed coupling

(Here $\sigma_X > 0$). Consider, for an adequate δ:

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} f(|X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|)\right]$$

Study the dynamic $d(X_t^{i,N} - \bar{X}_t^i)$ and $d(C_t^{i,N} - \bar{C}_t^i)$.
In the subspace $\{(X_t^{i,N} - \bar{X}_t^i) = 0\}$: deterministic contraction. $\Rightarrow$ Synchronous coupling
In the orthogonal subspace $\Rightarrow$ Reflection coupling (maximization of variance)

Idea: Non-synchronous coupling. Mixed reflection coupling and synchronous one.

$$dB_t^{i,X} - d\bar{B}_t^{i,X} = 2rc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X}$$

Strategy: Coupling method

Improved idea: Mixed coupling

Consider two white noises for each i: $B^{i,sc,X}$, $B^{i,rc,X}$.

Strategy: Coupling method

Improved idea: Mixed coupling

Consider two white noises for each i: $B^{i,sc,X}$, $B^{i,rc,X}$.
Consider two functions sc and $rc : \mathbb{R}^+ \to [0,1]$ such that $rc^2 + sc^2 = 1$:

Strategy: Coupling method

Improved idea: Mixed coupling

Consider two white noises for each i: $B^{i,sc,X}$, $B^{i,rc,X}$.
Consider two functions sc and $rc : \mathbb{R}^+ \to [0,1]$ such that $rc^2 + sc^2 = 1$:

$$dB_t^{i,X} = sc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,sc,X} + rc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X} \qquad d\bar{B}_t^{i,C} = dB_t^{i,C}$$

$$d\bar{B}_t^{i,X} = sc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,sc,X} - rc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X}$$

Then:

$$dB_t^{i,X} - d\bar{B}_t^{i,X} = 2rc\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X}$$

Strategy: Coupling method

Improved idea: Mixed coupling

Two Lyapunov functions:

1) Sufficient to have good bounds on moments of Z_t^i and $\bar{Z}_t^i$.

$$H(z) = \frac{1}{2}\gamma x^2 + \beta x + \frac{1}{2}c^2 + \alpha c + H_0$$

Strategy: Coupling method

Improved idea: Mixed coupling

Two Lyapunov functions:
1) Sufficient to have good bounds on moments of Z_t^i and $\bar{Z}_t^i$.

$$H(z) = \frac{1}{2}\gamma x^2 + \beta x + \frac{1}{2}c^2 + \alpha c + H_0$$

2) Need better control of the noise:

$$\tilde{H}(z) = \int_0^{H(z)} \exp\left(a\sqrt{u}\right) du = \frac{2}{a^2} \exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)} - 1\right) + \frac{2}{a^2}.$$

Strategy: Coupling method

Improved idea: Mixed coupling

Two Lyapunov functions:
2) Need better control of the noise:

$$\tilde{H}(z) = \int_0^{H(z)} \exp\left(a\sqrt{u}\right) du = \frac{2}{a^2} \exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)} - 1\right) + \frac{2}{a^2}.$$

We define new "distance":

$$\rho\left((z_j, z_j')_{1\leq j\leq N}\right) = \frac{1}{N}\sum_{i=1}^{N} f\left(r\left(z_i, z_i'\right)\right)\left(1 + \epsilon\tilde{H}\left(z_i\right) + \epsilon\tilde{H}\left(z_i'\right) + ...\right)$$

with concave f.

Strategy: Coupling method

Improved idea: Mixed coupling

With this coupling and this distance, we work on the dynamic of $e^{ct}\rho\left((Z_t^{j,N}, \bar{Z}_t^j)_{1\leq j\leq N}\right)$.
Good choice of parameters + contraction in various spaces + Gronwall's lemma $\Rightarrow$ Second theorem

State of the art: Mixed coupling

- Lindvall, Rogers (1986). *"Coupling of Multidimensional Diffusions by Reflection"*
- Eberle, Zimmer (2016). *"Sticky couplings of multidimensional diffusions with different drifts"*
  - Discuss the approach of sticky couplings
  - Provide total variation bounds
  - Prove the contraction of process on a linear subspace
- Eberle, Guillin, Zimmer (2019). *"Couplings and quantitative contraction rates for Langevin dynamics"*
  - applied to Langevin equation without interaction
- Pre-print: Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché (2021). *"Convergence Rates for the Vlasov-Fokker-Planck Equation and Uniform in Time Propagation of Chaos in Non Convex Cases"*

Perspectives

- Constraints on Lipschitz constants really tough. Maybe we can reduce constraint?

Perspectives

- Constraints on Lipschitz constants really tough. Maybe we can reduce constraint?
- Other types of interaction

Perspectives

- Constraints on Lipschitz constants really tough. Maybe we can reduce constraint?
- Other types of interaction
- Environmental noise?

Perspectives

- Constraints on Lipschitz constants really tough. Maybe we can reduce constraint?
- Other types of interaction
- Environmental noise?
- Random parameters α_i, β_i, γ_i, ...

Perspectives

- Constraints on Lipschitz constants really tough. Maybe we can reduce constraint?
- Other types of interaction
- Environmental noise?
- Random parameters α_i, β_i, γ_i, ...
- Synchronization of neurons

Thank you!