Uniform in time propagation of chaos for some non convex or singular particle systems.

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Metastability, mean field particle systems and non linear processes
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Introduction

Idea

In a system of N exchangeable interacting particles, as N increases, two particles become more and more statistically independent.

Mark Kac introduced the terminology *Propagation of chaos* to describe this phenomenon.

There is of course a huge litterature : Sznitman, Méléard, ... to the more recent by Malrieu, Mouhot, Mischler, Hauray, Delarue, Tse, Szpruch, Lacker, Jabin, Wang, Delgadino, Carrillo, Pavliotis....

Formal limit of mean-Field SDE

N-particle system on the torus $\mathbb{T}^d$

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt.$$

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Limit as N tends to infinity ?

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$$dX_t^i = \sqrt{2} dB_t^i + K * \mu_t^N(X_t^i) dt,$$

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}.$$

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Limit as N tends to infinity ? Formally

$$\begin{cases} d\bar{X}_t = \sqrt{2} dB_t + K * \bar{\rho}_t(\bar{X}_t) dt, \\ \bar{\rho}_t = \text{Law}(\bar{X}_t). \end{cases}$$

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$$dX_t^i = \sqrt{2} dB_t^i + \frac{1}{N} \sum_{j=1}^{N} K(X_t^i - X_t^j) dt. \quad \text{(PS)}$$

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Liouville equations

For the particle system

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt$$

$$\longleftrightarrow$$

$$\partial_t \rho_t^N = -\sum_{i=1}^{N} \nabla_{x_i} \cdot \left(\left(\frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \right) \rho_t^N \right) + \sum_{i=1}^{N} \Delta_{x_i} \rho_t^N.$$

For the non linear equation

$$\begin{cases} d\bar{X}_t = \sqrt{2}dB_t + K * \bar{\rho}_t(\bar{X}_t)dt, \\ \bar{\rho}_t = \mathrm{Law}(\bar{X}_t). \end{cases} \longleftrightarrow \quad \partial_t \bar{\rho}_t = -\nabla \cdot (\bar{\rho}_t (K * \bar{\rho}_t)) + \Delta \bar{\rho}_t.$$

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Propagation of chaos

In a system of N interacting particles, as N increases, two particles become more and more statistically independent.

To quantify this "more and more", we compare the law of any subset of k particles within the N particles system to the law of k independent non-linear particles.

We will add the difficulty that we want **uniform in time control**!

The distance we will use to compare these laws will induce a strategy of proof :

- Wasserstein distance leads to coupling (Sznitman, Malrieu, ...)
- Energy/Entropy leads to more analytical proofs (with functional inequalities!) (Carrillo, Mc Cann, Villani, Mouhot, Mischler, Tugaut, Fournier, Jabin, Wang, Lacker, Delgadino, Carrillo, ...)
- weak norm and Lions derivatives calculus (Chassagneux, Szpruch, Tse, Delarue, Schlichting ...)

Definition

We denote, for any $k \leq N$

$$\rho_t^{k,N}(x_1,..,x_k) = \int_{\mathbb{T}^{(N-k)d}} \rho_t^N(x_1,..,x_N) dx_{k+1}...dx_N$$
$$\bar{\rho}_t^k = \bar{\rho}_t^{\otimes k}$$

Definition

Let μ and ν be two probability measures. We consider the Wasserstein distance for $p \in [1,2]$

$$W_p^p(\nu,\mu) := \inf_{X\sim\nu, Y\sim\mu} \mathbb{E}(|X-Y|^p)$$

Definition

Let μ and ν be two probability measures. We consider the rescaled relative entropy

$$\mathcal{H}_N(\nu,\mu) = \begin{cases} \frac{1}{N}\mathbb{E}_\mu\left(\frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}\right) & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Coupling approach

Non convex case

Here we consider

$$dX_t^i = \sqrt{2}dB_t^i - \nabla V(X_t^i)dt - \frac{1}{N}\sum_j \nabla W(X_t^i - X_t^j)dt$$

with for example

- $V(x) = |x|^2$ and W convex
- $V(x) = |x|^4 - |x|^2$ and $W(x) = \delta|x|^2$.

The uniformly strictly convex case (and even degenerately strictly convex) is done via **synchronous coupling** :
use the same Brownian motion for the linear and non linear particles system

$$d\bar{X}_t^i = \sqrt{2}dB_t^i - \nabla V(\bar{X}_t^i)dt - \nabla W * \rho_t(\bar{X}_t^i)dt$$

$$\begin{aligned}
d|X_t^i - d\bar{X}_t^i|^2 &= -2(X_t^i - \bar{X}_t^i).(\nabla V(X_t^i) - \nabla V(\bar{X}_t^i))dt \\
&\quad -\frac{2}{N}(X_t^i - \bar{X}_t^i)\sum_j(\nabla W(X_t^i - X_t^j) - \nabla W * \rho_t(\bar{X}_t^i))dt \\
&= 2(X_t^i - \bar{X}_t^i).(\nabla V(X_t^i) - \nabla V(\bar{X}_t^i)d)t \\
&\quad -\frac{2}{N}(X_t^i - \bar{X}_t^i)\sum_j(\nabla W(X_t^i - X_t^j) - \nabla W(\bar{X}_t^i - \bar{X}_t^j))dt \\
&\quad -\frac{2}{N}(X_t^i - \bar{X}_t^i)\sum_j(\nabla W(\bar{X}_t^i - \bar{X}_t^j) - \nabla W * \rho_t(\bar{X}_t^i))dt.
\end{aligned}$$

After summing : use convexity for the first term, smallness of δ and Lipschizianity for the second one, moment control and law of large numbers for the last one ! Conclude by Gronwall's lemma so that

$$\sup_{t\geq 0} W_2^2(\rho_t^{1,N}, \bar{\rho}_t) \leq \frac{C}{N}.$$

The non convex case is trickier : synchronuous coupling does not help where you loose convexity...

One has to do reflection coupling : dB_t^i are replaced by

$$\left(I_d - 2(\bar{X}_t^i - X_t^i)(\bar{X}_t^i - X_t^i)^T / \|\bar{X}_t^i - X_t^i\|^2 \right) dB_t^i$$

but

- you cannot work with W_2 ! and even W_1 : necessary that we have a concave function of the distance for Itô's formula to help

$$W_{l^1,f}(\nu,\mu) := \inf_{X\sim\nu, Y\sim\mu} \mathbb{E} \frac{1}{N} \sum_y f(|X^i - Y^i|),$$

- put the reflection coupling on the linear particles in order to use again the LLN for the non linear particles !
- do tedious calculations for the choice of the concave function f

At the end :

$$\sup_{t\geq 0} W_1(\rho_t^{1,N}, \bar{\rho}_t) \leq \frac{C}{\sqrt{N}}.$$

Advantages/Defaults

1. does not need the confinement and interaction to be gradient
2. constructive but... perturbative (difficult to use interaction for uniformity in time)
3. extends to kinetic type models (Vlasov-Fokker-Planck equation)
4. for singular models... yes, for Dyson's Brownian motion (see recent work by G-Le Bris-Monmarché)
5. kinetic singular ?

Analytical approach

the 2D vortex model

The Biot-Savart kernel, defined in $\mathbb{T}^2$by

$$K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right).$$

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Consider the 2D incompressible Navier-Stokes system on $u \in \mathbb{T}^2$

$$\begin{aligned}\partial_t u =& - u \cdot \nabla u - \nabla p + \Delta u\\ \nabla \cdot u =& 0,\end{aligned}$$

where p is the local pressure. Taking the curl of the equation above, we get that $\omega(t,x) = \nabla \times u(t,x)$ satisfies

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Goal : Obtain a limit "$\rho_t^N \to \bar{\rho}_t$" as N tends to infinity for this Biot-Savart kernel.

Results

Theorem (Jabin-Wang ('18))

Under some assumptions (satisfied by the Biot-Savart kernel) there are positive constants C_1 and C_2 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq e^{C_1 t} \left(\mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_2}{N} \right)$$

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Theorem

Under some assumptions (satisfied by the Biot-Savart kernel) there are positive constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq C_1 e^{-C_2 t} \mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_3}{N}$$

Various distances

Corollary

Under some assumptions (satisfied by the Biot-Savart kernel), assuming moreover that $\rho_0^N = \bar{\rho}_0^N$*, there is a constant C such that for all* $k \leq N \in \mathbb{N}$ *and all* $t \geq 0$*,*

$$\|\rho_t^{k,N} - \bar{\rho}_t^k\|_{L^1} + \mathcal{W}_2\left(\rho_t^{k,N}, \bar{\rho}_t^k\right) \leq C\left(\left\lfloor \frac{N}{k} \right\rfloor\right)^{-\frac{1}{2}}$$

Step one : Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N), \quad \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left| \nabla_{x_i} \log \frac{\rho_t^N}{\bar{\rho}_t^N} \right|^2 d\mathbf{X}^N.$$

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It has been shown, by Jabin-Wang, that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_N(t) \leq &- \mathcal{I}_N(t) \\ &- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N \\ &- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(\operatorname{div} K(x_i - x_j) - \operatorname{div} K * \bar{\rho}_t(x_i)\right) d\mathbf{X}^N.\end{aligned}$$

Assumptions ?

$$\textbf{Goal :} \ K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

- $\bar{\rho} \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$

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- $\bar{\rho} \in \mathcal{C}^{\infty}(\mathbb{R}^+ \times \mathbb{T}^d)$ and there is $\lambda > 1$, s.t $\frac{1}{\lambda} \leq \bar{\rho} \leq \lambda$

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- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{T}^d)$
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Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.

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We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar\rho_t^N)\,, \quad \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left|\nabla_{x_i} \log \frac{\rho_t^N}{\bar\rho_t^N}\right|^2 d\mathbf{X}^N.$$

It has been shown, by Jabin-Wang, that

$$\frac{d}{dt}\mathcal{H}_N(t) \le -\mathcal{I}_N(t)$$
$$- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_t^N d\mathbf{X}^N$$
$$- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(\operatorname{div} K(x_i - x_j) - \operatorname{div} K * \bar\rho_t(x_i)\right) d\mathbf{X}^N.$$

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Step two : Integration by part

We are left with

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Idea : Use the regularity of $\bar{\rho}$ to deal with the singularity of K

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Idea : Use the regularity of $\bar{\rho}$ to deal with the singularity of K
Remark : Notice that, for the Biot-Savart kernel on the whole space $\mathbb{R}^2$

$$\tilde{K}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2},$$

we have $\tilde{K} = \nabla \cdot \tilde{V}$ with

$$\tilde{V}(x) = \frac{1}{2\pi}\begin{pmatrix} -\arctan\left(\frac{x_1}{x_2}\right) & 0 \\ 0 & \arctan\left(\frac{x_2}{x_1}\right) \end{pmatrix}.$$

Assumptions ?

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Justifying the calculations

- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{T}^d)$
  $\Longrightarrow \bar{\rho} \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (???)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^{\infty}$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^{d} \partial_\beta V_{\alpha,\beta}$ (Phuc-Torres '08).

Step two : Integration by part

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$A_N(t) := \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(V(x_i - x_j) - V * \bar{\rho}(x_i)\right) : \frac{\nabla^2_{x_i}\bar{\rho}_t^N}{\bar{\rho}_t^N} d\mathbf{X}^N$$

$$B_N(t) := \frac{1}{N}\sum_{i}\int_{\mathbb{T}^{dN}} \rho_t^N \frac{\left|\nabla_{x_i}\bar{\rho}_t^N\right|^2}{|\bar{\rho}_t^N|^2} \left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}(x_i)\right|^2 d\mathbf{X}^N.$$

Note that we would prefer to deal with the non linear particles which are i.i.d.

Step three : Change of reference measure and large deviation estimates

Lemma

For two probability densities μ and ν on a set Ω, and any $\Phi \in L^\infty(\Omega)$, $\eta > 0$ and $N \in \mathbb{N}$,

$$\mathbb{E}^\mu \Phi \leq \eta \mathcal{H}_N(\mu, \nu) + \frac{\eta}{N} \log \mathbb{E}^\nu e^{N\Phi/\eta}.$$

Large deviation estimates -1

Theorem (Jabin-Wang '18)

Consider any probability measure μ on $\mathbb{T}^d$, $\epsilon > 0$ and a scalar function $\psi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with $\|\psi\|_{L^\infty} < \frac{1}{2\epsilon}$ and such that for all $z \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} \psi(z, x)\mu(dx) = 0$. Then there exists a constant C such that

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{j_1, j_2=1}^{N} \psi(x_1, x_{j_1})\psi(x_1, x_{j_2})\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq C,$$

where C depends on

$$\alpha = (\epsilon\|\psi\|_{L^\infty})^4 < 1\ ,\ \ \beta = \left(\sqrt{2\epsilon}\|\psi\|_{L^\infty}\right)^4 < 1.$$

Large deviation estimates -2

Theorem (Jabin-Wang '18)

Consider any probability measure μ *on* $\mathbb{T}^d$ *and* $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ *with*

$$\gamma := \left(1600^2 + 36e^4\right)\left(\sup_{p\geq 1} \frac{\| \sup_z |\phi(\cdot, z)| \|_{L^p(\mu))}}{p}\right)^2 < 1.$$

Assume that ϕ *satisfies the following cancellations*

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x, z)\mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z, x)\mu(dx)\,.$$

Then, for all $N \in \mathbb{N}$*,*

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{i,j=1}^{N} \phi(x_i, x_j)\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq \frac{2}{1-\gamma} < \infty.$$

Conclusion

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq C\left(\mathcal{H}_N(t) + \frac{1}{N}\right) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$C = \hat{C}_1 \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda + \hat{C}_2 \|V\|_{L^\infty}^2 \lambda^2 d^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2$$

where $\hat{C}_1, \hat{C}_2$ are universal constants.

Step four : Uniform bounds and logarithmic Sobolev inequality

Two goals :

- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$

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Two goals :

- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$
- Uniform in time bounds on $\|\nabla \bar{\rho}_t\|_{L^\infty}$ and $\|\nabla^2 \bar{\rho}_t\|_{L^\infty}$

A logarithmic Sobolev inequality

Lemma (Tensorization)

If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, then for all N ≥ 0, $\nu^{\otimes N}$ satisfies a LSI with constant C_ν^{LS}

A logarithmic Sobolev inequality

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If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, then for all $N \geq 0$, $\nu^{\otimes N}$ satisfies a LSI with constant C_ν^{LS}

Lemma (Perturbation)

If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, and μ is a probability measure with density h with respect to ν such that, for some constant $\lambda > 0$, $\frac{1}{\lambda} \leq h \leq \lambda$, then μ satisfies a LSI with constant $C_\mu^{LS} = \lambda^2 C_\nu^{LS}$.

A logarithmic Sobolev inequality

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

For all $N \in \mathbb{N}$, $t \geq 0$ and all probability density $\mu_N \in \mathcal{C}^\infty_{>0}(\mathbb{T}^{dN})$,

$$\mathcal{H}_N\left(\mu_N, \bar{\rho}^N_t\right) \leq \frac{\lambda^2}{8\pi^2} \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{T}^d} \mu_N \left| \nabla_{x_i} \log \frac{\mu_N}{\bar{\rho}^N_t} \right|^2 d\mathbf{X}^N$$

Uniform in time bounds on the derivatives

Lemma

For all $n \geqslant 1$ and $\alpha_1, ..., \alpha_n \in [\![1, d]\!]$, there exist $C_n^u, C_n^\infty > 0$ such that for all $t \geqslant 0$,

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^\infty}^2 ds \leq C_n^\infty$$

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Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space H^m for all m, i.e in L^2

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

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$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar\rho_t\|_{L^2}^2 + \|\nabla\bar\rho_t\|_{L^2}^2 = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar\rho_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar\rho_t\|_{L^\infty}^2\|\nabla\bar\rho_t\|_{L^2}^2,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar\rho_t\|_{L^2}^2\|\nabla\bar\rho_t\|_{L^2}^2$$

$$+ \|K\|_{L^1}^2\|\bar\rho_t\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar\rho_t\|_{L^2}^2,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

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$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar{\rho}_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2\|\nabla\bar{\rho}_t\|_{L^2}^2 + \|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2,$$

etc

Assumptions ?

$$\textbf{Goal :}\ K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{T}^d)$ $\implies \bar{\rho} \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (? ? ?)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^{\infty}$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_{\alpha} = \sum_{\beta=1}^{d} \partial_{\beta} V_{\alpha,\beta}$ (Phuc-Torres '08).

Uniformity in time

- For all $n \geq 1$, $C^0_n := \|\nabla^n \bar{\rho}_0\|_{L^{\infty}} < \infty$
- $\|K\|_{L^1} < \infty$ (also used to show regularity).

Step five : Conclusion

There are constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ with $\int_0^t C_2(s)ds \le C_2^\infty$ for all $t \ge 0$ such that for all $t \ge 0$

$$\frac{d}{dt}\mathcal{H}_N(t) \le -(C_1 - C_2(t))\mathcal{H}_N(t) + \frac{C_3}{N}.$$

Multiplying by $\exp(C_1 t - \int_0^t C_2(s)ds)$ and integrating in time we get

$$\mathcal{H}_N(t) \le e^{-C_1 t + \int_0^t C_2(s)ds}\mathcal{H}_N(0) + \frac{C_3}{N}\int_0^t e^{C_1(s-t)+\int_s^t C_2(u)du}ds$$

$$\le e^{C_2^\infty - C_1 t}\mathcal{H}_N(t) + \frac{C_3}{C_1 N}e^{C_2^\infty},$$

which concludes.

On $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$

Everything works for regularized kernels K^ϵ, and the final result is independent of ϵ.

Assumptions

On the initial condition

- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
- For all $n \geq 1$, $C^0_n := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$

On the potential K

- $\|K\|_{L^1} < \infty$.
- In the sense of distributions, $\nabla \cdot K = 0$,
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$.

There are of course a lot of problems remaining

- up to now restricted to the torus... extend it to whole space ?
- logarithmic Sobolev inequalities for the invariant measure of the particles system ?
- 1-D Coulomb gaz, as in Cépa-Lépingle ?
- Vlasov-Fokker-Planck equation with singular kernel as in Bresch-Jabin-Soler's recent work ? (not uniform, no rate, ...)

Thank you