Exact simulation of the first time a diffusion process overcomes a given threshold

S. Herrmann

University of Burgundy, Dijon, France

joint work with Cristina ZUCCA (University of Turin)
and Nicolas MASSIN (University of Valenciennes)

May 19, 2022

Introduction

Simulation of random variables depending on the paths of a one-dimensional diffusion process: a challenging task.

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x < L.$$

Aim: Generation of different variates can be considered:

- the value X_t at a fixed time $t > 0$.
- the first passage time (FPT) through a given threshold

$$\tau_L := \inf\{t \geq 0 : X_t = L\}, \quad x < L.$$

- the exit time (ET) of an interval

$$\tau_I := \inf\{t \geq 0 : X_t \notin I\}, \quad x \in]a, b[.$$

Applications in different fields: breaking times (reliability), times of ruin (insurance), neuroscience, barrier options (finance),...

Different tools: explicit expression of the pdf, approximation of the density, approximation of the stochastic process, rejection sampling...

Explicit expressions for the FPT τ_L or for the ET τ_I.

Standard Brownian case ($B_0 = 0$):

1. The optional stopping thm applied to $M_t = \exp\{\lambda B_t - \frac{1}{2}\lambda^2 t\}$ leads to

$$\mathbb{E}[e^{-\lambda\tau_L}] = e^{-\sqrt{2\lambda}L}, \quad \lambda \geq 0.$$

Inversion of the Laplace transform:

$$\mathbb{P}(\tau_L \in dt) = \frac{1}{\sqrt{2\pi t^3}}\ e^{-\frac{L^2}{2t}}dt, \quad t > 0.$$

> Hence $\tau_L \sim L^2/G^2$
>
> where $G \sim \mathcal{N}(0,1)$.
>
> Easy and exact simulation !

2. Concerning τ_I with $I = [-1, 1]$, we know that:

$$p_\tau(t) = \sum_{n=0}^{+\infty}(-1)^n R_1(2n+1, t) \quad \text{with } R_1(n,t) := \frac{2n}{t^{3/2}}\ \phi\Big(\frac{n}{\sqrt{t}}\Big).$$

The following expansion also holds:

$$p_\tau(t) = \sum_{n=0}^{+\infty}(-1)^n R_2(2n+1, t) \quad \text{with } R_2(n,t) := \frac{\pi n}{2}\exp\Big(-\frac{n^2\pi^2}{8}\ t\Big).$$

- When the transition probability of (X_t) has an explicit expression...

Voltera-type integral equation (see Buonocore, Nobile, Ricciardi)

The pdf $f_L(t)$ of the FPT τ_L satisfies a Voltera-type equation depending on the probability current of the diffusion process.

Closed form results for the Brownian motion and for the O-U process.
In general: numerical approximation of the integral...

- General method: time discretization (Euler scheme).

$$X_{(n+1)\Delta} = X_{n\Delta} + \Delta b(X_{n\Delta}) + \sqrt{\Delta}\,\sigma(X_{n\Delta})G_n, \quad n \geq 0.$$

τ_L^Δ the FPT of the **discrete-time process**: we often observe an overestimation of the FPT.

1. a shift of the boundary (Broadie-Glasserman-Kou, Gobet-Menozzi)
2. computation of the probability for a Brownian bridge to hit the boundary during a small time interval (Giraudo-Saccerdote-Zucca)

Acceptance-rejection sampling: an exact simulation of the FPT

Principal idea: Let f and g two probability distribution functions, such that $h(x) := f(x)/g(x)$ is upper-bounded by a constant $c > 0$.

Aim: simulation of X with pdf f.

1. Generate a rv Y with pdf g.
2. Set $X = Y$ with conditional probab. $h(Y)/c$ otherwise go back to 1.

For any positive measure function ψ:

$$\mathbb{E}[\psi(X)] = \int_{\mathbb{R}} \psi(x)f(x)\,dx = \int_{\mathbb{R}} \psi(x)h(x)g(x)\,dx = \mathbb{E}[\psi(Y)h(Y)]$$

Important: h should be bounded and explicit!

Not quite so simple: h is related to a series in particular situations.

The aim is to use this general procedure for specific variables:

- the diffusion value X_t at time t (Beskos & Roberts, 2005)
- the stopping times τ_L (FPT) and the exit time τ_I

Remark: Lamperti's transform ⇒ simpler diffusion process

$$dX_t = dB_t + b(X_t)\,dt, \quad X_0 = x.$$

Regular drift b. Set $\beta(x) = \int_0^x b(y)dy$ and $\gamma := \frac{b^2+b'}{2}$.

1st Case: simulation of X_t for a given time t (Beskos & Roberts).

Using Girsanov's transformation and Itô's lemma:

$$\mathbb{E}_x[\psi(X_t)] = \mathbb{E}\left[\psi(x+B_t)e^{\beta(x+B_t)-\int_0^t \gamma(x+B_s)\,ds}\right] = \mathbb{E}[\psi(Y)h(Y)]$$

It permits to use a rejection sampling for Y whose distribution satisfies

$$g(y) := \frac{1}{\sqrt{2\pi t}\cdot g(\mathbb{R})}\, e^{\beta(y)-\frac{(y-x)^2}{2t}}, \quad \text{if } g(\mathbb{R}) < \infty,$$

associated with the weigth of acceptance given by

$$h(y) := g(\mathbb{R})\cdot\mathbb{E}\left[e^{-\int_0^t \gamma(x+B_s)\,ds}\middle| x+B_t = y\right] = g(\mathbb{R})\cdot\mathbb{E}\left[e^{-\int_0^t \gamma(b_s^{x\to y})\,ds}\right].$$

Here $(b_s^{x\to y}, 0 \le s \le t)$ stands for a Brownian bridge starting with the value x and ending in y at time t.

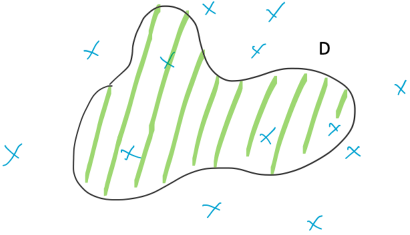
Proposal random variable: Y with p.d.f. $g(y) := \frac{1}{\sqrt{2\pi t}\cdot g(\mathbb{R})} e^{\beta(y) - \frac{(y-x)^2}{2t}}$ accepted with the weight proportional to $\mathbb{E}\left[e^{-\int_0^t \gamma(b_s^{x\to y})\, ds}\right]$.

Intuitive algorithm:

1. Generate Y with density g –> y
2. Generate a path of a Brownian bridge –> $b^{x\to y}$.
3. Accept y with probability weight proportional to $e^{-\int_0^t \gamma(b_s^{x\to y})\, ds}$. –> *area under the curve*

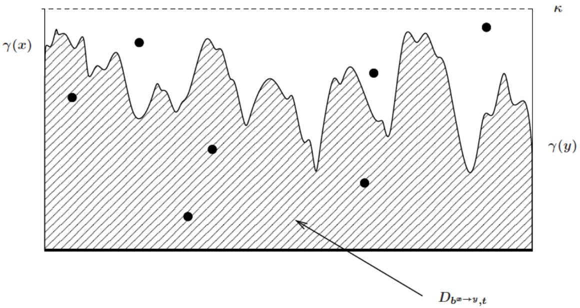
From now on, hyp: $0 \le \gamma(\cdot) \le \kappa$.

Poisson Process N with intensity λ, the Lebesgue measure on $\mathbb{R}^2$.

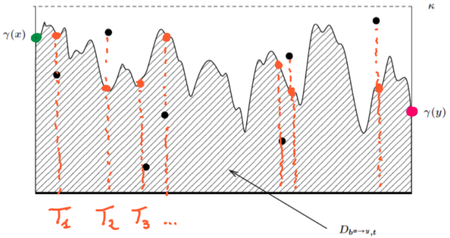


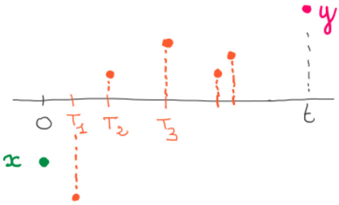
$\mathbb{P}[N(D) = 0] = e^{-\lambda(D)}$.

How to accept y with probability $\propto e^{-\int_0^t \gamma(b_s^{x\to y})\,ds}$? (BR)



- Generate N a P.P. on $[0,t]\times\mathbb{R}_+$, independent of the Brownian bridge.
- If $N(D_{b^{x\to y},t})=0$ then accept y.





Exact simulation of X_t – Algorithm $(BR)_t$

$(Y_n)_{n\geq 1}$ i.i.d. with density g, $(G_n)_{n\geq 1}$ i.i.d. $\mathcal{N}(0,1)$, $(E_n)_{n\geq 1}$ i.i.d. $\mathcal{E}(\kappa)$, $(U_n)_{n\geq 1}$ i.i.d. $\mathcal{U}([0,1])$. All sequences are independent.

Initialization: $k=0$, $n=0$.

Step 1. Set $k \leftarrow k+1$ then $Z=x$, $W=Y_k$ and $\mathcal{T}=0$.

Step 2. While $\mathcal{T}<t$ do:

- set $n \leftarrow n+1$
- $Z \leftarrow Z + \frac{E_n}{t-\mathcal{T}}\, W + \sqrt{\frac{E_n(t-\mathcal{T}-E_n)_+}{t-\mathcal{T}}}\, G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T}+E_n, t)$
- If $(\mathcal{T}<t$ and $\kappa\, U_n < \gamma(Z))$ then go to Step 1.

Outcome: the random variable W.

Theorem (Beskos-Roberts) Under suitable hyp., the outcome W of Algorithm $(BR)_t$ and the diffusion value X_t are identically distributed.

2nd Case: simulation of τ_L with L fixed (H. & Zucca, 2019).

Let us recall that $X_0 = x$,

$$dX_t = dB_t + b(X_t)\,dt,\ \beta(x) = \int_0^x b(y)dy \text{ and } \gamma := \frac{b^2+b'}{2}.$$

Combining the Girsanov transform and Itô's lemma permits to obtain:

$$\mathbb{E}[\psi(\tau_L^X)1_{\{\tau_L^X<\infty\}}] = \mathbb{E}\Big[\psi(\tau_L^B)\exp\Big(\int_0^{\tau_L^B} b(B_s)dB_s - \frac{1}{2}\int_0^{\tau_L^B} b^2(B_s)ds\Big)\Big]$$

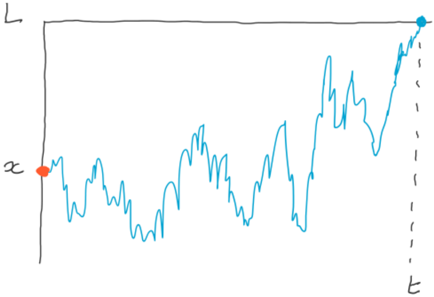
$$= \mathbb{E}\Big[\psi(\tau_L^B)e^{\beta(L)-\beta(x)}e^{-\int_0^{\tau_L^B}\gamma(B_s)\,ds}\Big] = \mathbb{E}[\psi(\tau_L^B)h(\tau_L^B)]$$

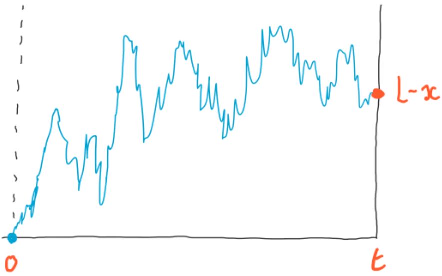
with

$$h(t) \propto \mathbb{E}\Big[e^{-\int_0^t \gamma(B_s)\,ds}\Big| B_0 = x,\ \tau_L^B = t\Big] = \mathbb{E}\Big[\exp - \int_0^t \gamma(L - R_s)ds\Big].$$

Here $(R_t,\ t \geq 0)$ stands for a Bessel bridge of dimension 3 starting in 0 and ending with the value $L - x$ at time t.

Why is a Bessel process appearing in the computations ?





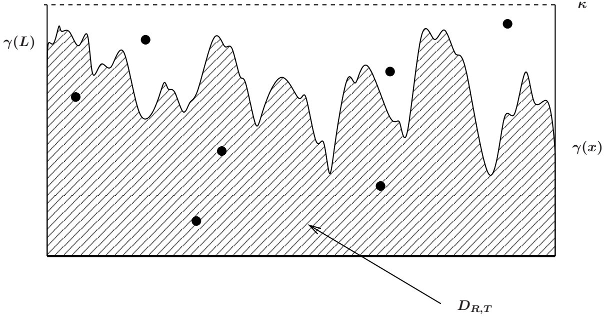
After a rotation of 180^o...

Heuristic algorithm for the generation of τ_L under the condition $0 \le \gamma(x) \le \kappa$.

Step 1: *Generate* $T = (L - x)^2 / G^2$ *with* $\boldsymbol{G \sim \mathcal{N}(0,1)}$.

Step 2: *Generate a Bessel bridge of dim* 3.

$$D_{R,T} := \Big\{ (t, v) \in [0, T] \times \mathbb{R}_+ : \ v \le \gamma(L - R_t) \Big\}.$$



Step 3: *Generate a P.P. N on* $[0, T] \times \mathbb{R}_+$, *indep. of the Bessel process.*

Step 4: *If* $N(D_{R,T}) = 0$ *then accept* T *otherwise go to Step 1.*

Exact simulation of τ_L – Algorithm (HZ)

$(G_n)_{n\geq 1}$ i.i.d. $\mathcal{N}_3(0,\mathrm{Id})$, $(e_n)_{n\geq 0}$ i.i.d $\mathcal{E}(\kappa)$, $(V_n)_{n\geq 1}$ i.i.d $\mathcal{U}([0,1])$, $(g_n)_{n\geq 1}$ i.i.d. $\mathcal{N}(0,1)$. All sequences are independent.

Initialization: $k=0$, $n=0$.

Step 1. $k \leftarrow k+1$, $\delta = (0,0,0)$, $\mathcal{W}=0$, $\mathcal{T}_k \leftarrow (L-x)^2/g_k^2$, $\mathcal{E}_0 = 0$ and $\mathcal{E}_1 = e_n$.

Step 2. While $\mathcal{E}_1 \leq \mathcal{T}_k$ do:

- set $n \leftarrow n+1$
- $\delta \leftarrow \frac{\mathcal{T}_k-\mathcal{E}_1}{\mathcal{T}_k-\mathcal{E}_0}\delta + \sqrt{\frac{(\mathcal{T}_k-\mathcal{E}_1)(\mathcal{T}_k-\mathcal{E}_0)}{\mathcal{T}_k-\mathcal{E}_0}}\ G_n$
- If $\kappa V_n \leq \gamma(L- \parallel \mathcal{E}_1(L-x)(1,0,0)/\mathcal{T}_k + \delta \parallel)$ then $\mathcal{W} \leftarrow 1$ else $\mathcal{W} \leftarrow 0$
- $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ and $\mathcal{E}_1 \leftarrow \mathcal{E}_1 + e_n$

Step 3. If $\mathcal{W}=0$ then $\mathcal{Y} \leftarrow \mathcal{T}_k$ otherwise go to *Step 1.*

Outcome: the random variable $\mathcal{Y}$.

Theorem

Under suitable conditions, the outcome $\mathcal{Y}$ of Algorithm (HZ) and τ_L are identically distributed.

Efficiency of the algorithm.

Rem.: Be carefull with the generation of the PP: if you sample all points, their averaged number is $\mathbb{E}[\kappa T] = \infty$: efficiency to be improved!

Number of iterations (step 1): $\mathbb{E}[\mathcal{I}] \leq \exp((L-x)\sqrt{2\kappa})$.

- Concerning $(L-x)$, linearization using space splitting.
- Concerning κ: if $0 < \gamma_0 \leq \gamma(x) \leq \kappa$ for all $x \in \mathbb{R}$,
  then replace $\gamma(\cdot) \leftarrow \gamma(\cdot) - \gamma_0$, $\kappa \leftarrow \kappa - \gamma_0$ & introduce the generation of $IG\left(\frac{L-x}{\sqrt{2\gamma_0}}, (L-x)^2\right)$ *(Michael-Schucany-Haas).*

Hyp. on γ, the average number of points used during the first iteration:

$$\mathbb{E}[\mathcal{N}_1] \leq M_{\gamma,1} + \kappa M_{\gamma,2}(x^2 + (L-x)^{(1+r)/2}), \quad \text{for } x < L.$$

Examples of generalization and numerics

Example. $dX_t = (2 + \sin(X_t))\, dt + dB_t,\ X_0 = 0$. We have $0 \leq \gamma \leq 5$.

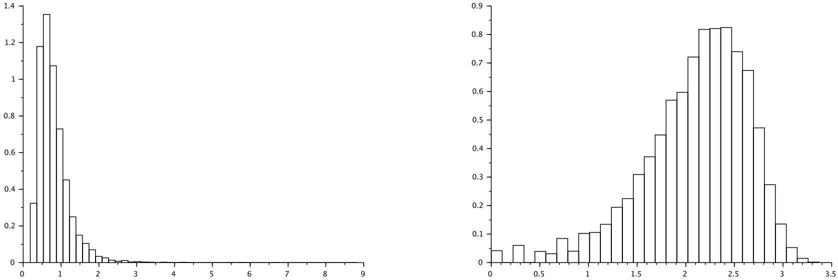


Figure: Histogram of the hitting time distribution for 10 000 simulations corresponding to the level $L = 2$ and starting position $X_0 = 0$ (left), histogram of the number of iterations in Algorithm (A1) in the $\log_{10}$-scale (right).

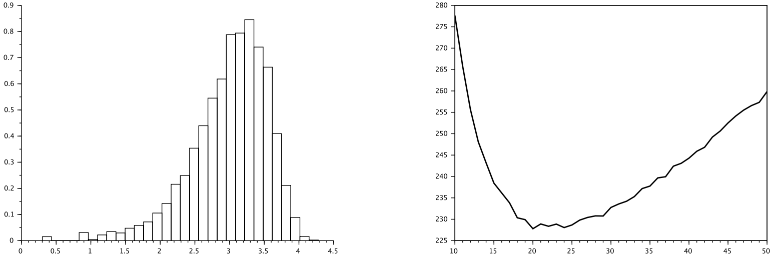


Figure: Number of random variables used in Algorithm (HZ-1) for 10 000 simulations with $L = 2$, $X_0 = 0$ in the $\log_{10}$-scale (left) and averaged number of random variables used in Algorithm (HZ-1) versus the number of slices k with $X_0 = 0$ and $L = 5$. The averaging uses 10 000 simulations.

Stopped diffusion processes:

- The algorithm (HZ) presented so far permits to observe τ_L and consequently the event $\tau_L < \mathbb{T}$ for $\mathbb{T}$ any fixed time.
- Another algorithm (A) permits to generate the conditional distribution of

$$X_{\mathbb{T}} \quad \text{given} \quad \tau_L > \mathbb{T}.$$

Algo (A) based on:

1. Exact generation of the Brownian motion $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$. Pdf:

$$f_{\mathbb{T}}(x) = \frac{1}{\sqrt{\mathbb{T}}} \frac{\phi(x/\sqrt{\mathbb{T}}) - \phi((x-2L)/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}, \quad x < L.$$

2. Rejection sampling: Girsanov's transform in a similar way as (HZ).

Combining Algo (HZ) (generation of τ_L) and (A) (conditional distribution of $X_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$) permits to generate the first time a jump diffusion overcomes a given threshold L.

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dB_t + \int_{\mathcal{E}} j(t, X_{t-}, v) p_\lambda(dv \times dt), \quad t \geq 0.$$

- $p_\lambda(dv \times dt)$ is a Poisson measure on $\mathcal{E} \times [0, T]$ whose intensity measure is given by $\lambda(dv)dt$, λ being non negative finite.
- the jump rate corresponds to $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \to \mathbb{R}$

We build a new algorithm which generates $(\tau_L \wedge \mathbb{T}, X_{\tau_L \wedge \mathbb{T}})$

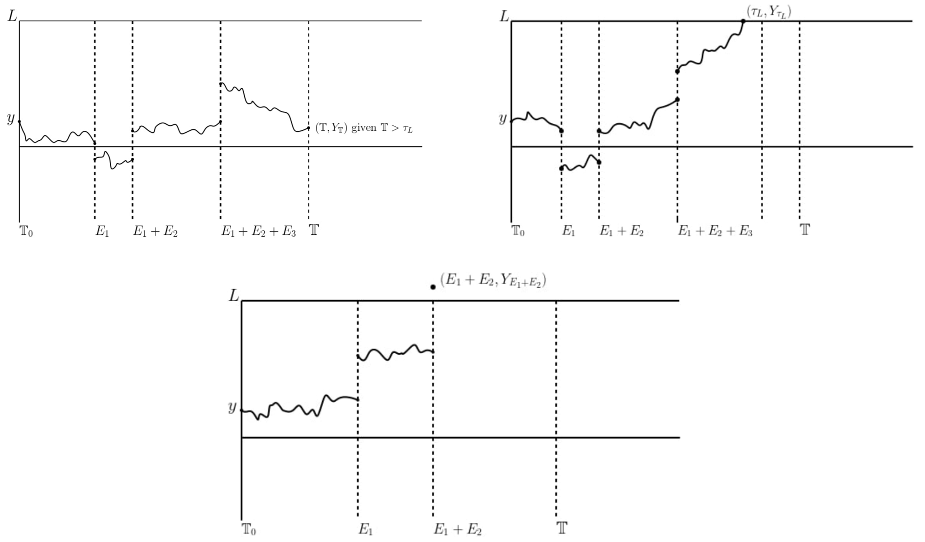


Figure: Three typical paths representing different scenarios

3rd Case: simulation of τ_I for a given $I = [a, b]$ (H. & Zucca, 2020).

We recall that $X_0 = x$,

$$dX_t = dB_t + b(X_t)\,dt,\ \beta(x) = \int_0^x b(y)dy \text{ and } \gamma := \frac{b^2 + b'}{2}.$$

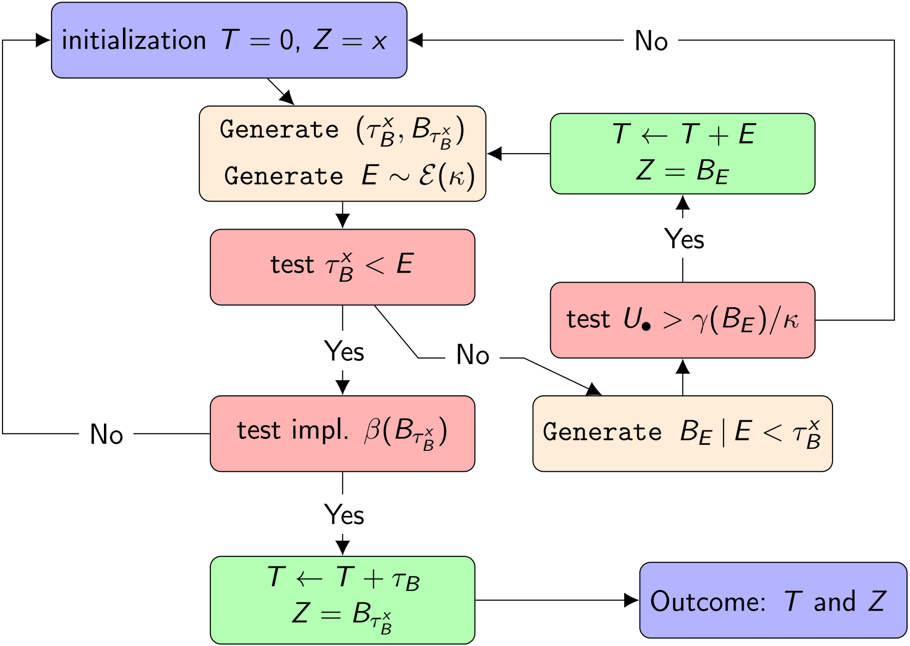
Using Girsanov's transformation and Itô's lemma:

$$\begin{aligned}\mathbb{E}[\psi(\tau_I^X, X_{\tau_I^X})] &= \mathbb{E}\left[\psi(\tau_I^B, X_{\tau_I^B}) e^{\beta(B_{\tau_I^B}) - \beta(x)} e^{-\int_0^{\tau_I^B} \gamma(B_s)\,ds}\right] \\ &= \mathbb{E}[\psi(\tau_I^B, B_{\tau_I^B}) h(\tau_I^B, B_{\tau_I^B})]\end{aligned}$$

with

$$\begin{aligned}h(t, y) &\propto e^{\beta(y)} \mathbb{E}\left[e^{-\int_0^t \gamma(B_s)\,ds} \middle| B_0 = x,\ \tau_I^B = t, B_{\tau_I^B} = y\right] \\ &= e^{\beta(y)} \mathbb{E}\left[e^{-\int_0^t \gamma(\xi_s)\,ds}\right]\end{aligned}$$

where $(\xi_s,\ 0 \leq s \leq t)$ is a constrained Brownian motion.


initialization $T = 0$, $Z = x$
No
Generate $(\tau_B^x, B_{\tau_B^x})$
Generate $E \sim \mathcal{E}(\kappa)$
$T \leftarrow T + E$
$Z = B_E$
Yes
test $\tau_B^x < E$
test $U_\bullet > \gamma(B_E)/\kappa$
Yes
No
test impl. $\beta(B_{\tau_B^x})$
No
Generate $B_E \mid E < \tau_B^x$
Yes
$T \leftarrow T + \tau_B$
$Z = B_{\tau_B^x}$
Outcome: T and Z

$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x \in (a, b),$$

Theorem

- If $\gamma(\cdot)$ is a non negative function on $[a, b]$ and upper bounded by κ, then the outcome of the algorithm (Z, T) has the same distribution as (X_{τ_I}, τ_I).
- Moreover the global cost is given by $\mathcal{N}_{\mathrm{tot}}^x$ satisfying::

$$\mathbb{E}[\mathcal{N}_{\mathrm{tot}}^x] \leq C(t_c, t_e) \cosh\left(\sqrt{\frac{\kappa}{2}}(b-a)\right), \quad \forall x \in]a, b[.$$

Generalization:

1. to any drift term $b \in \mathcal{C}^1([a, b])$. The modified algorithm is based on an iterative procédure.
2. to any diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ using the Lamperti transform.

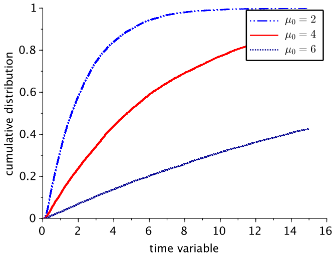


Figure: Empirical cumulative distribution function on $[0, 15]$ for the OU exit time of $[-1, 1]$ (10 000 simulations)

$$dX_t = -\mu_0 X_t dt + dB_t$$

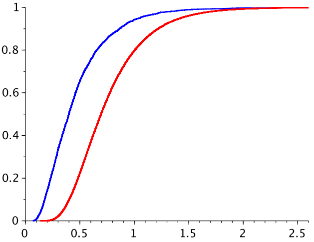


Figure: FET from $[a, b] = [-1, 2]$ for the diffusion

$$dX_t = (2 + \sin(X_t))\, dt + dB_t$$

(sample size: 100 000).

To sum up...

1. Exact simulation of the first passage time for a continuous diffusion.
2. Exact simulation of the first time a jump diffusion overcomes L
3. Exact simulation of the first exit time for continuous diffusion

Related questions:

- Exit time from a domain in $\mathbb{R}^d$ with $d \geq 2$.
- Exit time for nonlinear diffusions ???

S. H. and C. Zucca, *Exact simulation of the first-passage time of diffusions* J. Sci. Comput. 79 (2019), no. 3, 1477-1504.

S. H. and C. Zucca, *Exact simulation of first exit times for one-dimensional diffusion processes*, ESAIM M2NA 54 (2020), no.3, 811–844

S. H. and C. Zucca, *Exact simulation of diffusion first exit times: algorithm acceleration.* J. Mach. Learn. Res. 23 (2022)

S. H. and N. Massin, *Exact simulation of the first passage time through a given level for jump diffusions* (2021) arXiv:2106.05560