Vanishing noise limit of the first collision and the first collision-location between two self-stabilizing diffusions

a.k.a. A Kramers' type law for the collision time of two self-interacting diffusion processes and of their related particle approximation

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Aims: Given $\sigma, \alpha > 0$ and the two independent self-stabilizing diffusion processes:

$$X_t = x_1 + \sigma B_t - \int_0^t \left\{ \nabla V(X_s) + \alpha\big(X_s - \mathbb{E}[X_s]\big) \right\} ds, \quad t \geq 0,$$

$$Y_t = x_2 + \sigma \tilde{B}_t - \int_0^t \left\{ \nabla V(Y_s) + \alpha(Y_s - \mathbb{E}[Y_s]) \right\} ds, \quad t \geq 0,$$

establish the $\sigma \downarrow 0$-asymptotic of

- the first collision-time between X and Y, "given" by $C(\sigma) = \inf\{t \geq 0 : X_t = Y_t\}$,
- the first collision-location $X_{C(\sigma)} (= Y_{C(\sigma)})$.

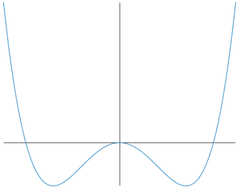
Same questions for the related particle systems:

$$X_t^{i,N} = x_1 + \sigma B_t^i - \int_0^t \left\{ \nabla V(X_s^{i,N}) + \alpha\left(X_s - \frac{1}{N}\sum_{j=1}^N X_s^{j,N} \right) \right\} ds, \quad t \geq 0,\ 1 \leq i \leq N,$$

$$Y_t^{i,N} = x_2 + \sigma \tilde{B}_t^i - \int_0^t \left\{ \nabla V(Y_s^{i,N}) + \alpha\left(Y_s^{i,N} - \frac{1}{N}\sum_{j=1}^N Y_s^{j,N} \right) \right\} ds, \quad t \geq 0,\ 1 \leq i \leq N.$$

Prototypical case

Double wells landscape (Kramers '40, Dawson '83): $d = 1$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$.



At the limit $\sigma = 0$,

$$X_t \longrightarrow \phi(t, x_1) = x_1 - \int_0^t V'(\phi(s, x_1))\, ds,$$

$$Y_t \longrightarrow \phi(t, x_2) = x_2 - \int_0^t V'(\phi(s, x_2))\, ds.$$

If $x_1 < -1$ and $1 < x_2$, at $\sigma = 0$ then collision(s) between X and Y can happen due to the action of the Brownian motions.

Heuristic: As $\sigma \downarrow 0$, it is expected that $C(\sigma)$ grows to ∞ at a certain rate, and $X_{C(\sigma)}$ should "persist" in a certain point between the wells $\lambda_1 = -1$ and $\lambda_2 = 1$.

General assumptions

(A) $-$ (0) B and $\tilde{B}$ are two independent $\mathbb{R}^d$-Brownian motions;

(A) $-$ (i) $V : \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^2$, locally Lipschitz, convex at infinity (namely $\inf_{|x|\geq R'} \nabla^2 V(x)$ is positive definite for some $R' > 0$) and is such that ∇V grows at a $2n$-polynomial rate on $\mathbb{R}^d$:

$$\sup_{x\in\mathbb{R}^d} \left\{(1+|x|^{2n})^{-1}|\nabla V(x)|\right\} < \infty\,.$$

(A) $-$ (ii) V admits two distinct (strict) local minima located in λ_1 and λ_2.

(A) $-$ (iii) Synchronization condition: α is large enough so that $\alpha I_d + \nabla^2 V$ positive definite.

(A) $-$ (iv) The starting points x_1 and x_2 lie in a respective basin of attraction of V:

$$\phi(t,x_i) = x_i - \int_0^t \nabla V(\phi(s,x_i))\, ds \underset{t\to\infty}{\longrightarrow} \lambda_i,\ i = 1,2.$$

Notes:

◦ **(A)** $-$ (iii) $\Rightarrow x \mapsto V(x) + \frac{\alpha}{2}|x-m|^2$ is strictly convex, for any $m \in \mathbb{R}^d$.

◦ **(A)** $-$ (iv) $\Rightarrow$ No collision at $\sigma = 0$.

General motivation

Stochastic Cucker-Smale model:

$$\begin{cases} dX_t^{i,N} = U_t^{i,N}\,dt, \\ dU_t^{i,N} = \displaystyle\sum_{j=1}^{N} \xi_{i,j}^N(t, X_t^{1,N}, \cdots, X_s^{N,N}) \left(U_t^{i,N} - U_t^{j,N} \right) \, dt + \sigma(X_t^{i,N}, U_t^{i,N})\, dW_t^i. \end{cases}$$

• $\sigma = 0$: Cucker-Smale model for flocking ([CK07]): Under certain conditions on the interaction/communication rate $\xi_{i,j}^N$, emergence of a common behaviour at large-time:

$$\lim_{t\to\infty} \max_{i,j} |U_t^{i,N} - U_t^{j,N}| = 0, \quad \lim_{t\to\infty} \max_{i,j} |X_t^{i,N} - X_t^{j,N}| < \infty.$$

• $\sigma \neq 0$: [CDP18]: Different types of noise (additive, multiplicative, idiosyncratic, common, ...) leads to differents probabilistic interpretation of the flocking ($L^p(\Omega)$, a.s., in proba., weak, ...)

Cucker-Smale models with "emerging" leaders and noise-induced collision: Two groups of populations each flocking as $t \to \infty$ around a prescribed attractor. Initially, no communication/interaction between the two populations, but a random perturbation forces the two population to interact with each others. As the effect of this perturbation vanishes, what happens of the collision point ?

Self-dtabilizing diffusions : Toy models where self-interacting diffusions represents overdamped/Kramers-Smoluchowski limit to the Langevin type models above, where possible leaders are static and possible post-collision effects are neglected.

One-dimensional case versus multidimensional case

• For $d = 1$, true collision happen and

$$C(\sigma) = \inf\{t \geq 0 \,:\, X_t = Y_t\}$$

is finite a.s.

• Whenever $d > 1$, B and $\tilde{B}$ do not necessarily hit each others, and without further conditions on V, $C(\sigma)$ is not well-defined $\Rightarrow$ Require an enlargement of the collision.

Enlarged first collision-time/enlarged first collision-location:

$$C_\varepsilon(\sigma) = \inf\{t \geq 0 \,:\, |X_t - Y_t| \leq 2\varepsilon\}, \qquad (X_{C_\varepsilon(\sigma)}, Y_{C_\varepsilon(\sigma)}).$$

Threshold:

$$\epsilon < \varepsilon_0 := 2^{-1} \inf_{t \geq 0} \left|\phi(t, x_1) - \phi(t, x_2)\right|$$

In particular, $\epsilon < \inf\{|x_1 - x_2|, |\lambda_1 - \lambda_2|\}$

Collision-time as an exit-time

$$C_\epsilon(\sigma) = \inf\{t \geq 0 \,:\, (X_t, Y_t) \notin (\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle_\epsilon\},$$

where

$$\triangle_\varepsilon := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, |x - y| = 2\varepsilon\}.$$

Freidlin-Wentzell's theory of gaussian perturbed exit time problem [FW98], [DZ10]

Gradient flow:

$$\Psi_t(z_0) = z_0 - \int_0^t \nabla U\left(\Psi_s(z_0)\right) ds\,,\, t \geq 0,$$

Perturbed version:

$$z_t^\sigma = z_0 + \sigma \mathcal{B}_t - \int_0^t \nabla U\left(z_s^\sigma\right) ds\,,\, t \geq 0.$$

Large Deviation Principle: For any finite time horizon T, and for any $\delta > 0$,

$$\lim_{\sigma \to 0} \frac{\sigma^2}{2} \log \mathbb{P}\left\{ \sup_{t \in [0;T]} |z_t^\sigma - \Psi_t(z_0)| > \delta \right\} = - \inf_{\Phi} \int_0^T \left| \frac{d\Phi}{dt} + \nabla U(\Phi(t)) \right|^2 dt,$$

where the supremum is taken over all possible $\mathcal{C}^1$-path Φ starting from z_0 and such that $\max_{0 \leq t \leq T} |\Phi(t) - \Psi_t(x_0)| > \delta$.

Definition

We say that the domain $\mathcal{G} \subset \mathbb{R}^d$ is stable/positively invariant by $-\nabla U$ if for all $z_0 \in \mathcal{G}$ the orbit $\{\Psi_t(z_0)\,;\, t \in \mathbb{R}_+\}$ remains in $\mathcal{G}$.

Theorem

Let U be a convex $\mathcal{C}^2$ function and $\mathcal{G}$ be an open bounded set of $\mathbb{R}^d$, stable by ∇U. Assume also that for all $x_0 \in \partial\mathcal{G}$, $\Psi_t(x_0)$ converges to a unique point a_0 at large time. Then, for any z_0 in $\mathcal{G}$, and for

$$\tau_{\mathcal{G}}(\sigma) := \inf\{t \geq 0 \,:\, z_t^\sigma \notin \mathcal{G}\}\,,$$

we have:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H}-\delta\right)\right] < \tau_{\mathcal{G}}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H}+\delta\right)\right]\right\} = 1\,,$$

where $\underline{H}$ is the exit-cost of $\mathcal{G}$:

$$\underline{H} = \inf_{z\in\partial\mathcal{G}} \inf_{T>0} \inf_{\Phi} \left\{I_{T,z}(\Phi)\right\} = \inf_{x\in\partial\mathcal{G}} \Big(U(x) - U(a_0)\Big)$$

Additionally, if $\inf_{x\in\partial\mathcal{G}}\Big(U(x)-U(a_0)\Big)$ is achieved in a unique point $z_\star$ in $\partial\mathcal{G}$, then, for all $\delta > 0$, $z_0 \in \mathcal{G}$,

$$\lim_{\sigma\to 0} \mathbb{P}\left\{|z^\sigma_{\tau_{\mathcal{G}}(\sigma)} - z_\star| < \delta\right\} = 1.$$

• Hermann, Imkeller and Peithmann [HIP08]: Kramers' type law for the self-stabilizing diffusion:

$$Z_t = y + \sigma B_t - \int_0^t \left\{ \nabla U(Z_s) + \int \nabla \phi(Z_s - y)\mu_s(dy) \right\} ds, \; \mu_t = \mathrm{Law}(Z_t), \qquad t \geq 0.$$

Assuming U and ϕ are relatively smooth and strictly convex functions, and $\mathcal{G}$ is a stable set,

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \exp\left[\frac{2}{\sigma^2} \left(\underline{H} - \delta \right) \right] < \tau_{\mathcal{G}}(\sigma) < \exp\left[\frac{2}{\sigma^2} \left(\underline{H} + \delta \right) \right] \right\} = 1 \,,$$

for

$$\underline{H} = \inf_{z \in \partial G} H,$$

$$H(z) = \big(U(z) + \phi(z - z_\star) - U(z_\star), \quad z_\star = \mathrm{argmin} U.$$

• Tugaut *et al.* 2007-2021: Kramers' type law in the case of a double wells landscape and other globally non-convex situations (e.g. [T21] for the case of the granular media equation).

Heuristic Kramers' type law for the first collision time between X and Y: If $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle_\varepsilon$ was stable,

$$\log(C_\epsilon(\sigma)) \approx \frac{2}{\sigma^2}\underline{\mathcal{H}}_\epsilon, \quad (X_{C_\epsilon(\sigma)}, Y_{C_\epsilon(\sigma)}) \approx \mathcal{M}_\varepsilon,$$

for

$$\underline{\mathcal{H}}_\varepsilon = \inf_{(x,y)\in\partial\triangle_\varepsilon} H(x,y), \quad \mathcal{M}_\varepsilon = \text{argmin}_{\partial\triangle_\varepsilon} H,$$

$$H(x,y) = V(x) - V(\lambda_1) + \frac{\alpha}{2}|x-\lambda_1|^2 + V(y) - V(\lambda_2) + \frac{\alpha}{2}|y-\lambda_2|^2.$$

For $\varepsilon \ll 1$, $\sigma \downarrow 0$,

$$\log(C_\epsilon(\sigma)) \approx \frac{2}{\sigma^2}\inf_x H(x,x), \quad (X_{C_\epsilon(\sigma)}, Y_{C_\epsilon(\sigma)}) \approx \text{argmin}_x H(x,x).$$

Heuristic Kramers' type law for the first collision time between X and Y: If $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle_\varepsilon$ was stable,

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$$\underline{\mathcal{H}}_\varepsilon = \inf_{(x,y)\in\partial\triangle_\varepsilon} H(x,y), \quad \mathcal{M}_\varepsilon = \text{argmin}_{\partial\triangle_\varepsilon} H,$$

$$H(x,y) = V(x) - V(\lambda_1) + \frac{\alpha}{2}|x-\lambda_1|^2 + V(y) - V(\lambda_2) + \frac{\alpha}{2}|y-\lambda_2|^2.$$

For $\varepsilon \ll 1$, $\sigma \downarrow 0$,

$$\log(C_\epsilon(\sigma)) \approx \frac{2}{\sigma^2}\inf_x H(x,x), \quad (X_{C_\epsilon(\sigma)}, Y_{C_\epsilon(\sigma)}) \approx \text{argmin}_x H(x,x).$$

Problem: No *stability* property of $(\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle_\varepsilon$.
Alternative:

$$\begin{aligned} C_\epsilon(\sigma) &= \inf_{\lambda \in \mathbb{R}^d} \beta_{\lambda,\epsilon}(\sigma), \\ \beta_{\lambda,\epsilon}(\sigma) &:= \inf\left\{t \geq 0 \,:\, (X_t, Y_t) \in B(\lambda,\epsilon) \times B(\lambda,\epsilon)\right\}. \end{aligned}$$

Startegy: (Coupling techniques *a la* J. Tugaut) 1/ Establish preliminaries Kramers' type laws on

$$x_t^\sigma = x_1 + \sigma B_t - \int_0^t \left\{\nabla V(x_s^\sigma) + \alpha(x_s^\sigma - \lambda_1)\right\} ds, \quad t \geq 0,$$

$$y_t^\sigma = x_2 + \sigma \tilde{B}_t - \int_0^t \left\{\nabla V(y_s^\sigma) + \alpha(y_s^\sigma - \lambda_2)\right\} ds, \quad t \geq 0.$$

2/ Transfer the Kramers' type laws to the self-stabilizing diffusions.

Preliminaries: First collision time of two stochastic gradient flows

Dynamics:

$$x_t^\sigma = x_1 + \sigma B_t - \int_0^t \nabla\Psi_1\left(x_s^\sigma\right) ds\,,\, t \geq 0,$$

$$y_t^\sigma = x_2 + \sigma \widetilde{B}_t - \int_0^t \nabla\Psi_2\left(y_s^\sigma\right) ds\,\,,\, t \geq 0.$$

Assumptions:

- Ψ_1 and Ψ_2 are of class $\mathcal{C}^2$, strictly convex and admits λ_1 and λ_2 as their respective minimizers.
- $\inf_{t\geq 0} |\phi_t^1(x_1) - \phi_t^2(x_2)| =: 2\epsilon_0 > 0$ for $\phi_t^i(z) = z - \int_0^t \nabla\Psi_i(\phi_s^i(z))\, ds,\ t \geq 0,\ i = 1, 2.$

Approximated first collision time:

$$c_\epsilon(\sigma) := \inf\left\{t \geq 0\ :\ |x_t^\sigma - y_t^\sigma| \leq 2\epsilon\right\},\ \epsilon < \epsilon_0.$$

Reformulation: $c_\epsilon(\sigma) = \inf_\lambda \beta_{\lambda,\epsilon}(\sigma)$,

$$\beta_{\lambda,\rho}(\sigma) \quad = \quad \inf\{t \geq 0\ :\ (x_t^\sigma, y_t^\sigma) \notin (\mathbb{R}^d \times \mathbb{R}^d) \setminus (B(\lambda,\epsilon) \times B(\lambda,\epsilon))\}$$

Additional enlargement: $\widehat{\beta}_{\lambda,\rho}(\sigma) = \inf\{t \geq 0\ :\ (x_t^\sigma, y_t^\sigma) \in \mathcal{D}^1_{\lambda,\epsilon} \times \mathcal{D}^2_{\lambda,\epsilon}\}$,

$$\mathcal{D}^i_{\lambda,\epsilon} := \{\phi^{i,+}(t, x_i)\ :\ t \geq 0,\ z \in B(\lambda,\rho)\}, \qquad \phi^{i,+}(t, x_i) = z + \int_0^t \nabla\Psi_i(\phi^{i,+}(s, x_i))\, ds,\ t \geq 0\,.$$

Advantages:

• Whenever the "prescribed" collision λ is far from the wells (i.e. $\min_{i=1,2}(|\lambda - \lambda_i|) > \epsilon$), the domain $\left(\mathbb{R}^d \setminus \mathcal{D}^1_{\lambda,\epsilon}\right) \times \left(\mathbb{R}^d \setminus \mathcal{D}^2_{\lambda,\epsilon}\right)$ is stable by $(-\nabla\Psi^1, -\nabla\Psi^2)$.

• Whenever λ is too close to one of the wells, say $|\lambda - \lambda_1| < \epsilon$ then $B(\lambda,\rho)$ is an attractive set for $\phi^{1,+}$ and $\mathcal{D}^1_{\lambda,\rho} = \mathbb{R}^d$, meanwhile $\mathbb{R}^d \setminus \mathcal{D}^2_{\lambda,\epsilon}$ is stable by $-\nabla\Psi^2$).

• The case $\{|\lambda - \lambda_1| = \epsilon\} \cup \{|\lambda - \lambda_2| = \epsilon\}$ is singular and requires a slight rescaling.

Last approximation: For $0 < \rho < 1$,

$$\widehat{\beta}^{\rho}_{\lambda,\epsilon}(\sigma) := \inf\left\{t \geq 0 \,:\, (x^\sigma_t, y^\sigma_t) \in \mathcal{O}_{\lambda,\epsilon,\rho}\right\} \quad (1)$$

where the domain $\mathcal{O}_{\lambda,\epsilon,\rho}$ is given by

$$\mathcal{O}_{\lambda,\epsilon,\rho} := \begin{cases} \mathcal{D}^1(\lambda,\rho\epsilon) \times \mathcal{D}^2(\lambda,\epsilon) \text{ if } |\lambda - \lambda_1| = \epsilon, \\ \mathcal{D}^1(\lambda,\epsilon) \times \mathcal{D}^2(\lambda,\rho\epsilon) \text{ if } |\lambda - \lambda_2| = \epsilon, \\ \mathcal{D}^1(\lambda,\epsilon) \times \mathcal{D}^2(\lambda,\epsilon) \text{ otherwise.} \end{cases}$$

• Applying Freidlin-Wentzell's exit-time estimates:

Lemma

For any λ *in* $\mathbb{R}^d$ *and for any* $\delta > 0$,

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \exp \left[\frac{2}{\sigma^2} \left(\widehat{h}^{\rho}_{\epsilon}(\lambda) - \delta \right) \right] < \widehat{\beta}^{\rho}_{\lambda,\epsilon}(\sigma) < \exp \left[\frac{2}{\sigma^2} \left(\widehat{h}^{\rho}_{\epsilon}(\lambda) + \delta \right) \right] \right\} = 1 \,,$$

for

$$\widehat{h}^{\rho}_{\epsilon}(\lambda) = \begin{cases} \inf_{x \in \partial B(\lambda;\rho\epsilon)} (\Psi_1(x) - \Psi_1(\lambda_1)) + \inf_{y \in \partial B(\lambda;\epsilon)} (\Psi_2(y) - \Psi_2(\lambda_2)) \; \textit{if} |\lambda - \lambda_1| = \epsilon, \\ \inf_{x \in \partial B(\lambda;\epsilon)} (\Psi_1(x) - \Psi_1(\lambda_1)) + \inf_{y \in \partial B(\lambda;\rho\epsilon)} (\Psi_2(y) - \Psi_2(\lambda_2)) \; \textit{if} |\lambda - \lambda_2| = \epsilon, \\ \inf_{x \in \mathcal{D}^1_{\lambda,\epsilon}} (\Psi_1(x) - \Psi_1(\lambda_1)) + \inf_{y \in \mathcal{D}^2_{\lambda,\epsilon}} (\Psi_2(y) - \Psi_2(\lambda_2)) \; \textit{otherwise}, \end{cases}$$

Moreover, we have: for any $\delta > 0$,

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \min \left(dist\Big(x^{\sigma}_{\widehat{\beta}^{\rho}_{\lambda,\epsilon}(\sigma)}, B(\lambda,\epsilon)\Big), dist\Big(y^{\sigma}_{\widehat{\beta}^{\rho}_{\lambda,\epsilon}(\sigma)}, B(\lambda,\epsilon)\Big) \right) \leq \delta \right\} = 1 \,,$$

for $dist(x, B(\lambda,\epsilon)) := \inf_{z \in B(\lambda,\epsilon)} |x - z|$.

- From the domains $\mathcal{D}^1_{\lambda,\epsilon}$ and $\mathcal{D}^2_{\lambda,\epsilon}$ to $B(\lambda,\epsilon)$:

Lemma

The same Kramers' type law holds for

$$\beta^{\rho}_{\lambda,\epsilon}(\sigma) = \begin{cases} \inf\{t \geq 0 \,:\, (x^\sigma_t, y^\sigma_t) \in B(\lambda,\rho\epsilon) \times B(\lambda,\epsilon)\} & \text{if} |\lambda - \lambda_1| = \epsilon, \\ \inf\{t \geq 0 \,:\, (x^\sigma_t, y^\sigma_t) \in B(\lambda,\epsilon) \times B(\lambda,\rho\epsilon)\} & \text{if} |\lambda - \lambda_2| = \epsilon, \\ \inf\{t \geq 0 \,:\, (x^\sigma_t, y^\sigma_t) \in B(\lambda,\epsilon) \times B(\lambda,\epsilon)\} & \text{otherwise.} \end{cases}$$

- Asymptotic $\rho = 1$: $\widehat{h}^1_\epsilon(\lambda) := \lim_{\rho\to 1} \widehat{h}^\rho_\epsilon(\lambda)$.

Lemma

For any $\lambda \in \mathbb{R}^d$, and for any $\delta > 0$:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\widehat{h}^1_\epsilon(\lambda) - \delta\right)\right] < \beta_{\lambda,\epsilon}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\widehat{h}^1_\epsilon(\lambda) + \delta\right)\right]\right\} = 1\,.$$

Moreover,

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\min\left(\mathit{dist}\Big(x^\sigma_{\beta_{\lambda,\epsilon}(\sigma)}, B(\lambda,\epsilon)\Big), \mathit{dist}\Big(y^\sigma_{\beta_{\lambda,\epsilon}(\sigma)}, B(\lambda,\epsilon)\Big)\right) \leq \delta\right\} = 1\,.$$

• Kramers' type laws for $c_\epsilon(\sigma) = \inf_\lambda \beta_{\lambda,\epsilon}(\sigma) = \inf\{t \geq 0 \,:\, |x_t^\sigma - y_t^\sigma| = \epsilon\}$:

Preliminary remark:

$$\widehat{h}_\epsilon^1(\lambda) = \begin{cases} \inf_{x \in \partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) + \inf_{y \in \partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big) \text{ if } \min_{i=1,2} |\lambda - \lambda_i| \geq \epsilon, \\ \inf_{y \in \partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big) \text{ if } |\lambda - \lambda_1| < \epsilon\,, \\ \inf_{x \in \partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) \text{ if } |\lambda - \lambda_2| < \epsilon\,. \end{cases}$$

Choosing $\epsilon < \epsilon_c$ for

$$\epsilon_c := \inf\left\{\epsilon \in (0, \epsilon_0) \,:\, \inf_{\lambda \in B(\lambda_i,\epsilon)} \inf_{z \in \partial B(\lambda,\epsilon)} \big(\Psi_j(z) - \Psi_j(\lambda_j)\big) = \inf_\lambda h_\epsilon(\lambda),\ i \neq j \in \{1,2\}\right\},$$

then

$$\inf_\lambda \widehat{h}_\epsilon^1(\lambda) \approx \inf_\lambda h_\epsilon(\lambda),$$

$$h_\epsilon(\lambda) := \inf_{x \in \partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) + \inf_{y \in \partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big).$$

• Kramers' type laws for $c_\epsilon(\sigma) = \inf_\lambda \beta_{\lambda,\epsilon}(\sigma) = \inf\{t \geq 0 \,:\, |x_t^\sigma - y_t^\sigma| = \epsilon\}$:

Preliminary remark:

$$\widehat{h}_\epsilon^1(\lambda) = \begin{cases} \inf\limits_{x\in\partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) + \inf\limits_{y\in\partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big) \text{ if } \min\limits_{i=1,2} |\lambda - \lambda_i| \geq \epsilon, \\ \inf\limits_{y\in\partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big) \text{ if } |\lambda - \lambda_1| < \epsilon\,, \\ \inf\limits_{x\in\partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) \text{ if } |\lambda - \lambda_2| < \epsilon\,. \end{cases}$$

Choosing $\epsilon < \epsilon_c$ for

$$\epsilon_c := \inf\left\{\epsilon \,\in\, (0,\epsilon_0) \,:\, \inf_{\lambda\in B(\lambda_i,\epsilon)} \inf_{z\in\partial B(\lambda,\epsilon)} \big(\Psi_j(z) - \Psi_j(\lambda_j)\big) = \inf_\lambda h_\epsilon(\lambda),\ i \neq j \in \{1,2\}\right\},$$

then

$$\inf_\lambda \widehat{h}_\epsilon^1(\lambda) \approx \inf_\lambda h_\epsilon(\lambda),$$

$$h_\epsilon(\lambda) := \inf_{x\in\partial B(\lambda,\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) + \inf_{y\in\partial B(\lambda,\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big).$$

Theorem

For any $\epsilon < \epsilon_c$ *and* $\delta > 0$*:*

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \exp\left[\frac{2}{\sigma^2}\left(\underline{h}_\epsilon - \delta\right)\right] < c_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{h}_\epsilon + \delta\right)\right]\right\} = 1\,,$$

where

$$\underline{h}_\epsilon = \inf_\lambda h_\epsilon(\lambda).$$

In addition, for $\mathcal{H}_\epsilon$ *the set of all minimizers* λ_ϵ *of* h_ϵ*:*

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \inf_{\lambda_\epsilon \in \mathcal{H}_\epsilon} \max\left(dist\big(x^\sigma_{c_\epsilon(\sigma)}, B(\lambda_\epsilon, \epsilon)\big), dist\big(y^\sigma_{c_\epsilon(\sigma)}, B(\lambda_\epsilon, \epsilon)\big)\right) \le \delta \right\} = 1\,.$$

Note: The exit-cost $\underline{h_\epsilon}$ can be achieved in more than one point, but, as $\epsilon \downarrow 0$,

$$h_\epsilon(\lambda) \to \big(\Psi_1(\lambda) - \Psi_1(\lambda_1)\big) + \big(\Psi_2(\lambda) - \Psi_2(\lambda_2)\big),$$

and $\mathcal{H}_\epsilon$ concentrates onto a single point, the minimizer of h_0.

Corollary

For

$$h_0(\lambda) = \big(\Psi_1(\lambda) - \Psi_1(\lambda_1)\big) + \big(\Psi_2(\lambda) - \Psi_2(\lambda_2)\big),$$

$$\lambda_0 = argmin h_0 = (\nabla\Psi_1 + \nabla\Psi_2)(0)^{-1},$$

and $\underline{h_0} := h_0(\lambda_0)$*, we have: for any* $\delta > 0$*:*

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{h}_0 - \delta\right)\right] < c_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{h}_0 + \delta\right)\right]\right\} = 1$$

and

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{\max\left(|x^\sigma_{c_\epsilon(\sigma)} - \lambda_0|, |y^\sigma_{c_\epsilon(\sigma)} - \lambda_0|\right) \leq \delta\right\} = 1\,.$$

Kramers' type law for the first collision time between two self-stabilizing diffusions

Under the assumptions $(A)-(i)$ to $(A)-(iv)$, [HIP08]: the dynamics $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are wellposed (in the pathwise sense) and

$$\sup_{t\geq 0}\mathbb{E}[|X_t|^p+|Y_t|^p]<\infty,\ \forall p\in\mathbb{Z}.$$

Moreover, [T21]: For any $\kappa>0$, there exists a finite time T_κ and a critical diffusion σ_κ such that

$$\max_{\sigma\leq\sigma_\kappa, t\geq T_\kappa}\mathbb{E}[|X_t-\lambda_1|^2]+\mathbb{E}[|Y_t-\lambda_2|^2]\leq\kappa^2.$$

Corollary (Coupling estimate)

For any $\kappa>0$, there exists $T_\kappa>0$ such that

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\max_{t\geq T_\kappa}|X_t-x^\sigma_{t,T_\kappa}|+|Y_t-y^\sigma_{t,T_\kappa}|\geq\kappa\right\}=0.$$

for

$$\begin{aligned} x^\sigma_{t,T_\kappa} &= X_{T_\kappa}+\sigma(B_t-B_{T_\kappa})-\int_{T_\kappa}^t\{\nabla V(x^\sigma_s)+\alpha(x^\sigma_s-\lambda_1)\}\,ds,\\ y^\sigma_{t,T_\kappa} &= Y_{T_\kappa}+\sigma(\tilde{B}_t-\tilde{B}_{T_\kappa})-\int_{T_\kappa}^t\{\nabla V(y^\sigma_s)+\alpha(y^\sigma_s-\lambda_2)\}\,ds,\end{aligned}$$

From the Kramers' type law for (x^σ, y^σ) to the Kramers' law for (X, Y):

Theorem

Given $\epsilon > 0$ small enough,

$$H_\epsilon(\lambda) = \inf_{x \in \partial B(\lambda;\epsilon)} \big(\Psi_1(x) - \Psi_1(\lambda_1)\big) + \inf_{y \in \mathbb{B}(\lambda;\epsilon)} \big(\Psi_2(y) - \Psi_2(\lambda_2)\big),$$

$$\Psi_1(x) := V(x) + \frac{\alpha}{2}||x - \lambda_1||^2, \quad \Psi_2(y) := V(y) + \frac{\alpha}{2}||y - \lambda_2||^2,$$

and $\underline{H}_\epsilon = \inf H_\epsilon$, for any $\delta > 0$,

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H}_\epsilon - \delta\right)\right] < C_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H}_\epsilon + \delta\right)\right]\right\} = 1\,.$$

In addition, for $\mathcal{M}_\epsilon$ the set of all minimizers λ_ϵ of $\lambda \mapsto H_\epsilon(\lambda)$, and for ϵ small enough:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\inf_{\lambda_\epsilon \in \mathcal{M}_\epsilon} \max\Big(dist\big(X_{c_\epsilon(\sigma)}, B(\lambda_\epsilon, \epsilon)\big), dist\big(Y_{c_\epsilon(\sigma)}, B(\lambda_\epsilon, \epsilon)\big)\Big) \le \delta\right\} = 1\,.$$

Threshold: $\epsilon < \epsilon_c$ with

$$\epsilon_c = \inf\left\{\epsilon \le \varepsilon_0 \,:\, \inf_{\lambda \in \mathbb{B}(\lambda_i;\epsilon)} \inf_{z \in \partial\mathbb{B}(\lambda;\epsilon)} \left(V(z) - V(\lambda_j) + \frac{\alpha}{2}||z - \lambda_j||^2\right) = \inf_\lambda H_\epsilon(\lambda),\ i \ne j \in \{1, 2\}\right\}.$$

Theorem

For any $\delta > 0$, we have

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}-\delta\right)\right] < C_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}+\delta\right)\right]\right\} = 1\,.$$

where

$$\underline{H_0} = \min H_0(\lambda),$$

$$H_0(\lambda) = \lim_{\epsilon\to 0} H_\epsilon(\lambda) = 2V(\lambda) - V(\lambda_1) - V(\lambda_2) + \frac{\alpha}{2}|\lambda-\lambda_1|^2 + \frac{\alpha}{2}|\lambda-\lambda_2|^2.$$

Moreover, for

$$\lambda_0 := argmin_\lambda H_0(\lambda)\,,$$

it holds

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{|X_{C_\epsilon(\sigma)}-\lambda_0|\le\delta\right\} = 1 = \lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{|Y_{C_\epsilon(\sigma)}-\lambda_0|\le\delta\right\}\,.$$

Collision location:

$$\lambda_0 = \left(\nabla V + \alpha I_d\right)^{-1}(\alpha(\lambda_1+\lambda_2)/2)\,.$$

Example: For $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $\alpha > 1$,

$$\lambda_0 = 0, \quad \inf \underline{H}_0 = \alpha - 1\,,$$

$$\mathcal{M}_\epsilon = \left\{\alpha - 1, 4\epsilon^4 - 2\epsilon^2 + \alpha(2\epsilon\pm 1)^2 + \frac{1}{2}\right\}.$$

Collision time for the particle systems

Particle model:

$$X_t^{i,N} = x_1 + \sigma B_t^i - \int_0^t \left\{ \nabla V(X_s^{i,N}) + \alpha\big(X_s^{i,N} - \frac{1}{N}\sum_{j=1}^N X_s^{j,N}\big) \right\} ds, \quad t \geq 0,$$

$$Y_t^{i,N} = x_2 + \sigma \tilde{B}_t^i - \int_0^t \left\{ \nabla V(Y_s^{i,N}) + \alpha\big(Y_s^{i,N} - \frac{1}{N}\sum_{j=1}^N Y_s^{j,N}\big) \right\} ds, \quad t \geq 0,$$

for $(B_t^1)_{t\geq 0}, \cdots, (B_t^N)_{t\geq 0}$, and $(\tilde{B}_t^1)_{t\geq 0}, \cdots, (\tilde{B}_t^N)_{t\geq 0}$, two families of independent Brownian motions.

Approximated first collision-time:

$$C_\epsilon^{i,N}(\sigma) = \inf\left\{ t \geq 0 \,:\, |X_t^{i,N} - Y_t^{i,N}| \leq \epsilon \right\}, \, 1 \leq i \leq N.$$

Anticipated exit cost: As the potential related to each family of particles is given by

$$\Upsilon_N(\mathbf{x}^N) = \sum_{i=1}^N V(x_i) + \frac{\alpha}{2N}\sum_{i,j=1}^N |x_i - x_j|^2, \, \mathbf{x}^N = (x_1, \cdots, x_N) \in \mathbb{R}^{Nd},$$

the exit-cost related to the successive asymptotic "$\sigma \downarrow 0$ next $\epsilon \downarrow 0$" of $C_\epsilon^{i,N}(\sigma)$:

$$\inf_{\lambda\in\mathbb{R}^{dN}} \inf_{\mathbf{x}^N\in\partial B^{i,N}(\lambda,\epsilon)} \Upsilon_N(\mathbf{x}^N) - \Upsilon_N(\lambda_1, \cdots, \lambda_1) + \inf_{\lambda\in\mathbb{R}^{dN}} \inf_{\mathbf{y}^N\in\partial B^{i,N}(\lambda,\epsilon)} \Upsilon_N(\mathbf{y}^N) - \Upsilon_N(\lambda_2, \cdots, \lambda_2),$$

for

$$B^{i,N}(\lambda, \epsilon) = \left\{ \mathbf{x}^N = (x_1, \cdot, x_N) \in \mathbb{R}^{dN} \,;\, x_i \in B(\lambda, \epsilon) \right\}.$$

Preliminary note

• As long as (A) hold the particle systems and their "linear" analog:

$$x_{t,T}^{i,\sigma} = X_T^{i,N} + \sigma(B_t^i - B_T^i) - \int_T^t \nabla V(x_s^{i,\sigma})ds - \int_0^t \alpha(x_s^{i,\sigma} - \lambda_1)ds\,,\ T \leq t,$$

$$y_{t,T}^{i,\sigma} = Y_T^{i,N} + \sigma(\widetilde{B}_t^i - \widetilde{B}_T^i) - \int_T^t \nabla V(y_s^{i,\sigma})ds - \int_0^t \alpha(y_s^{i,\sigma} - \lambda_2)ds\,,\ T \leq t.$$

are well-posed in the pathwise sense. Moreover, for all $1 \leq i \leq N$, T finite and $1 \leq p < \infty$

$$\max_{t\in[0,T]} \mathbb{E}[|X_t^{i,N}|^p + |Y_t^{i,N}|^p] < \infty.$$

• Propagation of chaos: For $X^1, \cdots, X^N$ and $Y^1, \cdots, Y^N$, N independents copies of X_t and Y driven respectively by $B^1, \cdots, B^N$ and $\tilde{B}^1, \cdots, \tilde{B}^N$,

$$\mathbb{E}[\max_{0\leq t\leq T} |X_t^{i,N} - X_t^i|^p + \max_{0\leq t\leq T} |Y_t^{i,N} - Y_t^i|^2] \leq \frac{C(\sigma, T)}{N}.$$

Non-uniform propagation of chaos $\Rightarrow$ We cannot rely on the Kramers' law established in the mean-field limit situation to deal with the particle case.

$\Rightarrow$ **Start over and apply a strategy analog to the mean-field case**.

Lemma

For any $\kappa > 0$, there exists a (non-random) time $0 \leq T_\kappa$, uniform with respect to σ, and a (non-random) finite number of particles N_κ, both finite and independent of σ, such that, for all $N \geq N_\kappa$, it holds

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \max_{t \in \left[T_\kappa, \exp\left[\frac{2}{\sigma^2}(\underline{H}_\epsilon + 2)\right]\right]} \left(\left| \frac{1}{N} \sum_{j=1}^N X_t^{j,N} - \lambda_1 \right| + \left| \frac{1}{N} \sum_{j=1}^N Y_t^{j,N} - \lambda_2 \right| \right) \leq 2\kappa \right\} = 1 \,.$$

Proposition

For any $\xi > 0$, there exists $0 < T_\xi < \infty$ and $0 < N_\xi < \infty$ such that

$$\lim_{\sigma \to 0} \mathbb{P}\left(\sup_{T_\kappa \leq t \leq \exp\left[\frac{2}{\sigma^2}(\underline{H}_\epsilon + 2)\right]} \left\{ |X_t^{i,N} - x_{t,T_\kappa}^{i,\sigma}| + |Y_t^{i,N} - y_{t,T_\kappa}^{i,\sigma}| \right\} \geq \xi \right) = 0 \,.$$

Kramers' law for the first collision time

(Analogs to the mean-field limit - in particular same threshold $\epsilon < \epsilon_c$).

Theorem

Let $\mathcal{M}_\epsilon$ be the set of minimizers of H_ϵ. Then, for any $\delta > 0$ and ϵ small enough, provided that N is large enough:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \exp\left[\frac{2}{\sigma^2}\left(\underline{H_\epsilon} - \delta\right)\right] < C_\epsilon^{i,N}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_\epsilon} + \delta\right)\right]\right\} = 1\,,$$

and

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \inf_{\lambda_\epsilon \in \mathcal{M}_\epsilon} \max\left(dist(X^{i,N}_{C_\epsilon^{i,N}(\sigma)}, B(\lambda_\epsilon, \epsilon)), dist(Y^{i,N}_{C_\epsilon^{i,N}(\sigma)}, B(\lambda_\epsilon, \epsilon)) \right) \leq \delta \right\} = 1\,.$$

Corollary

For any $\delta > 0$, provided N is large enough:

$$\lim_{\epsilon \to 0} \lim_{\sigma \to 0} \mathbb{P}\left\{ \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} - \delta\right)\right] < C_\epsilon^{i,N}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} + \delta\right)\right]\right\} = 1\,.$$

Moreover, for any $1 \leq i \leq N$,

$$\lim_{\epsilon \to 0} \lim_{\sigma \to 0} \mathbb{P}\left\{ \max\left(|X^{i,N}_{\mathcal{C}_{\epsilon,N}(\sigma)} - \lambda_0|, |Y^{i,N}_{\mathcal{C}_{\epsilon,N}(\sigma)} - \lambda_0| \right) \leq \delta \right\} = 1\,.$$

Note on the one-dimensional case

In this situation, one can deal more directly with the true collision times:

$$C(\sigma) = \inf\left\{t \geq 0 \,:\, X_t = Y_t\right\}, \quad C^{i,N}(\sigma) = \inf\left\{t \geq 0 \,:\, X_t^{i,N} = Y_t^{i,N}\right\}.$$

Theorem

For any $\delta > 0$:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} - \delta\right)\right] < C(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} + \delta\right)\right]\right\} = 1\,.$$

Moreover, for λ_0 the minimizer of H_0,

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\left|X_{C(\sigma)} - \lambda_0\right| \leq \delta\right\} = 1.$$

Theorem

For any $\delta > 0$, and N sufficiently large:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} - \delta\right)\right] < C^{i,N}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} + \delta\right)\right]\right\} = 1\,,$$

and, for all $1 \leq i \leq N$

$$\lim_{\sigma \to 0} \mathbb{P}\left\{|X^{i,N}_{C^{i,N}(\sigma)} - \lambda_0| \leq \delta\right\} = 1.$$

Generalizations

Our main results holds true for:

- more general self-stabilizing forces:

$$X_t = x_1 + \sigma B_t - \int_0^t \left\{ \nabla V(X_s) + \nabla F\big(X_s - \mathbb{E}[X_s]\big) \right\} ds, \quad t \geq 0,$$

$$Y_t = x_2 + \sigma \tilde{B}_t - \int_0^t \left\{ \nabla V(Y_s) + \nabla F(Y_s - \mathbb{E}[Y_s]) \right\} ds, \quad t \geq 0,$$

provided that F is a smooth function such that $F(x) = G(|x|)$ where $G : \mathbb{R} \to \mathbb{R}$ is a even polynomial function G, with a degree larger than 2, satisfying $G(0) = 0$ (i.e. framework of [T20]). Same for the related particle systems.

- Multi-wells confining potential: For instance if V admits m wells located at $\lambda_1, \cdots, \lambda_m$ then, again, the Kramers'law for $C_\epsilon(\sigma)$, $C_{\epsilon,N}(\sigma)$, $C(\sigma)$ and $C_N(\sigma)$ hold and the collision λ_0 is located at

$$\Big(\sum_{l=1}^m \nabla \Psi_l\Big)^{-1}\Big(\alpha m^{-1} \sum_{l'=1}^m \lambda_{l'}\Big), \qquad \Psi_l(x) = V(x) + \frac{\alpha}{2}|x - \lambda_l|^2 .$$

- Random initial conditions: As long as (x_0^σ, y_0^σ) or (X_0, Y_0) are a.s. bounded (sufficient for control of moments), and the law of X_0 and Y_0 have full support on different basin of attraction of V, our main results for the self-stabilizing systems still hold true. Not so much for the particle systems.

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