An introduction to self-interacting diffusions

Aline Kurtzmann

Workshop May 2022

05/18/2022
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1 Generalities
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2. Self-attracting diffusion on $\mathbb{R}$ (with Victor Kleptsyn)
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3. Discretisation and dynamical system
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3. Discretisation and dynamical system
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5. Final result
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Brownian polymer

Durrett and Rogers (1992) on $\mathbb{R}^d$:

$$dX_t = dB_t + \int_0^t f(X_t - X_s) \, ds \, dt,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and bounded.
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Question: what is the normalisation of $X$?
Applications: physics, biology.
Generalities

Self-attracting case

Studied by:

- Cranston & Le Jan (1995): linear and $1 - d$ constant interaction ($f(x) = -a \, \text{sign}(x)$),

- Raimond (1997): constant interaction ($d \geq 2$, $f(x) = -a x / |x|$ with $a > 0$),


1) Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, decreasing and bounded. Suppose that there exists $C$, $\rho > 0$ and $k \in \mathbb{N}^*$ such that $|f(x)| \geq C e^{-\rho/|x|^k}$ around 0. Then $X_t$ converges a.s.

2) When the interaction is not local, $f(x) = -\text{sign}(x) 1_{\{|x| \geq a\}}$, then the trajectories remain bounded a.s. (but do not converge).

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- Herrmann & Roynette (2003):

**Theorem (Herrmann & Roynette, 2003)**

1) Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, decreasing and bounded. Suppose that there exists $C, \rho > 0$ and $k \in \mathbb{N}^*$ such that $|f(x)| \geq Ce^{-\rho/|x|^k}$ around 0. Then $X_t$ converges a.s.

2) When the interaction is not local, $f(x) = -\text{sign}(x)1_{\{|x| \geq a\}}$, then the trajectories remain bounded a.s. (but do not converge).
Generalities

Self-repulsive case

Theorem (Mountford & Tarrès, 2007)
Let \( f(x) = \frac{x}{1 + |x|^{1+\beta}} \) with \( 0 < \beta < 1 \). Then, there exists \( c > 0 \) such that with probability \( 1/2 \), \( \frac{X_t}{t^{\alpha}} \to c \), where \( \alpha = \frac{2}{1+\beta} \).
Generalities

Conjecture (Durrett and Rogers)

Theorem (Tarrès & Tóth & Valkó, 2012)
Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has a compact support, $xf(x) \geq 0$ and $f(-x) = -f(x)$. Then $\frac{X_t}{t}$ converges a.s. toward 0.
What is a self-interacting diffusion?

- Solution of

\[ dX_t = dB_t - F(t, X_t, \mu_t)dt \]

- \[ \mu_t = \frac{1}{t} \int_0^t \delta X_s ds \]
Reinforced diffusion on a compact set

Benaïm, Ledoux and Raimond (2002), Benaïm and Raimond (2003, 2005) on a compact manifold:

\[
dX_t = dB_t - \frac{1}{t} \int_0^t \nabla_x W(X_t, X_s) ds \, dt
\]
Reinforced diffusion on a compact set

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$$dX_t = dB_t - \frac{1}{t} \int_0^t \nabla_x W(X_t, X_s) ds dt$$

Heuristic: show that $\mu_t := \frac{1}{t} \int_0^t \delta_{X_s} ds$ is close to a deterministic flow.
Generalities  Study of self-interacting diffusions

Difficulty of the study on $\mathbb{R}^d$

Let

$$dX_t = dB_t - (\log t)^3 \nabla W(X_t - \bar{\mu}_t)dt, \; X_0 = x$$

where $\bar{\mu}_t = \frac{1}{t} \int_0^t X_s ds$.

Theorem (Chambeu & K)

1. The process $Y_t = X_t - \bar{\mu}_t$ converges a.s. to $Y_\infty$, where $Y_\infty$ belongs to the set of local minima of $W$. Moreover, for each local minimum $m$, we have $\mathbb{P}(Y_\infty = m) > 0$. 
Generalities

Study of self-interacting diffusions

Difficulty of the study on $\mathbb{R}^d$

Let

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Theorem (Chambeu & K)

1. The process $Y_t = X_t - \mu_t$ converges a.s. to $Y_\infty$, where $Y_\infty$ belongs to the set of local minima of $W$. Moreover, for each local minimum $m$, we have $\mathbb{P}(Y_\infty = m) > 0$.

2. On the set $\{Y_\infty = 0\}$, both $X_t$ and $\mu_t$ converge a.s. to $\mu_\infty := \int_0^\infty Y_s \frac{ds}{s}$. Moreover, on the set $\{Y_\infty \neq 0\}$, we have $\lim_{t \to \infty} X_t / \log t = Y_\infty$. 
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Study

\[
dX_t = dB_t - \left( V'(X_t) + \frac{1}{t} \int_0^t W'(X_t - X_s) \, ds \right) \, dt
\]

\[
= dB_t - \left( V'(X_t) + W' \ast \mu_t(X_t) \right) dt
\]

\[
\mu_t = \frac{1}{t} \int_0^t \delta_X s \, ds
\]
Study

\[ dX_t = dB_t - \left( V'(X_t) + \frac{1}{t} \int_0^t W'(X_t - X_s) \, ds \right) \, dt \]

\[ = dB_t - (V'(X_t) + W' * \mu_t(X_t)) \, dt \]

\[ \mu_t = \frac{1}{t} \int_0^t \delta_{X_s} \, ds \]

\[ V = 0. \]
Example: quadratic $W$

Lemma

Let $W(x) = ax^2$ with $a > 0$. Then a.s. the empirical measure $\mu_t$ converges (weakly) to $\mu_\infty$ where $\mu_\infty(\cdot - \bar{\mu}_\infty) \sim \mathcal{N}(0, 1/a)$ and $\bar{\mu}_\infty$ is also a Gaussian variable.
Example: quadratic $W$

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Let $W(x) = \frac{1}{2} (x - 1)^2$. Then $\bar{\mu}_t = \frac{1}{t} \int_0^t X_s \, ds$ and $X_t$ diverge a.s.
Set of hypotheses on the interaction potential (H)

- $W$ is $C^2$, strictly uniformly convex and symmetric,
Set of hypotheses on the interaction potential (H)

- $W$ is $C^2$, strictly uniformly convex and symmetric,
- there exist $C, k > 0$ such that

$$|W(x)| + |W'(x)| + |W''(x)| \leq C(1 + |x|^k).$$
Results

Theorem

Suppose that \( W \) satisfies the assumption (H). Then there exists a unique probability density function \( \rho_\infty \) such that a.s.

\[
\mu_t \rightarrow \rho_\infty (\cdot - c_\infty)dx.
\]
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Discretisation and dynamical system

Relation with a Markovian system

\[ \dot{\mu} = \Pi(\mu) - \mu, \]

where \[ \Pi(\mu) := \frac{1}{Z(\mu)} e^{-\frac{1}{2} W^* \mu}. \]
Relation with a Markovian system

\( \mu_t \) is asymptotically close to the deterministic dynamical system:

\[
\dot{\mu} = \Pi(\mu) - \mu,
\]

where \( \Pi(\mu) := \frac{1}{Z(\mu)} e^{-2W*\mu} \).
Strategy of the proof

Compare on $[T_n, T_{n+1}]$ the trajectories of

$$\begin{align*}
  dX_t &= dB_t - W' * \mu_t(X_t) dt \\
  dY_t &= dB_t - W' * \mu_{T_n}(Y_t) dt
\end{align*}$$

with those of the corresponding process where $\mu_t$ is replaced by $\mu_{T_n}$:
Strategy of the proof

- Compare on $[T_n, T_{n+1}]$ the trajectories of

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with those of the corresponding process where $\mu_t$ is replaced by $\mu_{T_n}$:

$$dY_t = dB_t - W' * \mu_{T_n}(Y_t) dt$$

- Estimate the speed of convergence of the empirical measure of $Y$ toward the invariant probability measure $\Pi(\mu_{T_n})$
Discretisation and dynamical system

Strategy for the approximation by a dynamical system

- Compare the flow obtained by the “Euler method”

\[ \mu[T_n, T_{n+1}] = \mu T_n + \frac{\Delta T_n}{T_{n+1}} \left( \mu[T_n, T_{n+1}] - \mu T_n + \text{error} \right) \]

with the flow

\[ \dot{\mu} = \frac{1}{T_n} (\Pi(\mu) - \mu) \]
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Definition

The center of the probability measure is the point $c_\mu$ such that

$$W' \ast \mu(c_\mu) = 0.$$  

We define the centered measure $\mu^c$ as

$$\mu^c(A) = \mu(A + c_\mu).$$
The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
- Lyapunov function: free energy
The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
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- Estimation of the speed of convergence (decrease of the entropy)
The deterministic system

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- Lyapunov function: free energy
- Estimation of the speed of convergence (decrease of the entropy)
- Convergence of the center
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Theorem

Suppose that $W$ satisfies the hypothesis $(H)$. Then:

1. there exists a unique probability density function $\rho_\infty$ centered such that a.s.

$$\mu_t^c \to \rho_\infty(x)dx,$$

2. a.s. the center $c_t = c(\mu_t)$ converges to a (random) limit $c_\infty$. 

Theorem

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   \[ \mu_t^c \to \rho_\infty(x)dx, \]

2. a.s. the center $c_t = c(\mu_t)$ converges to a (random) limit $c_\infty$. 