An introduction to self-interacting diffusions

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Outline

1. Generalities

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2. Self-attracting diffusion on $\mathbb{R}$ (with Victor Kleptsyn)

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2. Self-attracting diffusion on $\mathbb{R}$ (with Victor Kleptsyn)
3. Discretisation and dynamical system
4. Centering
5. Final result

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Brownian polymer

Durrett and Rogers (1992) on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \mathrm{d}B_t + \int_0^t f(X_t - X_s)\,\mathrm{d}s\,\mathrm{d}t,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is mesurable and bounded.

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Question: what is the normalisation of X?
Applications: physics, biology.

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- Herrmann & Roynette (2003):

Theorem (Herrmann & Roynette, 2003)

1) Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, decreasing and bounded. Suppose that there exists $C, \rho > 0$ and $k \in \mathbb{N}^*$ such that $|f(x)| \geq Ce^{-\rho/|x|^k}$ around 0. Then X_t converges a.s.
2) When the interaction is not local, $f(x) = -sign(x)\mathbf{1}_{\{|x| \geq a\}}$, then the trajectories remain bounded a.s. (but do not converge).

Self-repulsive case

Theorem (Mountford & Tarrès, 2007)

Let $f(x) = \frac{x}{1+|x|^{1+\beta}}$ *with* $0 < \beta < 1$*. Then, there exists* $c > 0$ *such that with probability 1/2,* $\frac{X_t}{t^\alpha} \to c$*, where* $\alpha = \frac{2}{1+\beta}$.

Conjecture (Durrett and Rogers)

Theorem (Tarrès & Tóth & Valkó, 2012)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ *has a compact support,* $xf(x) \geq 0$ *and* $f(-x) = -f(x)$. *Then* $\frac{X_t}{t}$ *converges a.s. toward 0.*

What is a self-interacting diffusion?

- Solution of

$$\mathrm{d}X_t = \mathrm{d}B_t - F(t, X_t, \mu_t)\mathrm{d}t$$

- $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$

Reinforced diffusion on a compact set

Benaïm, Ledoux and Raimond (2002), Benaïm and Raimond (2003, 2005) on a compact manifold:

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{t}\int_0^t \nabla_x W(X_t, X_s)\mathrm{d}s\,\mathrm{d}t$$

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Heuristic: show that $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s$ is close to a deterministic flow.

Difficulty of the study on $\mathbb{R}^d$

Let

$$\mathrm{d}X_t = \mathrm{d}B_t - (\log t)^3 \nabla W(X_t - \overline{\mu}_t)\mathrm{d}t,\ X_0 = x$$

where $\overline{\mu}_t = \frac{1}{t}\int_0^t X_s\,\mathrm{d}s$.

Theorem (Chambeu & K)

1. *The process $Y_t = X_t - \overline{\mu}_t$ converges a.s. to Y_∞, where Y_∞ belongs to the set of local minima of W. Moreover, for each local minimum m, we have $\mathbb{P}(Y_\infty = m) > 0$.*

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2. *On the set $\{Y_\infty = 0\}$, both X_t and $\overline{\mu}_t$ converge a.s. to $\overline{\mu}_\infty := \int_0^\infty Y_s \frac{ds}{s}$. Moreover, on the set $\{Y_\infty \neq 0\}$, we have $\lim_{t\to\infty} X_t / \log t = Y_\infty$.*

Outline

Study

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t - \left(V'(X_t) + \frac{1}{t}\int_0^t W'(X_t - X_s)\,\mathrm{d}s \right)\mathrm{d}t \\ &= \mathrm{d}B_t - (V'(X_t) + W' * \mu_t(X_t))\mathrm{d}t \end{aligned}$$

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$V = 0.$

Example: quadratic *W*

Lemma

Let $W(x) = ax^2$ with $a > 0$. Then a.s. the empirical measure μ_t converges (weakly) to μ_∞ where $\mu_\infty(\cdot - \bar{\mu}_\infty) \sim \mathcal{N}(0, 1/a)$ and $\bar{\mu}_\infty$ is also a Gaussian variable.

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Let $W(x) = \frac{1}{2}(x-1)^2$. Then $\overline{\mu}_t = \frac{1}{t}\int_0^t X_s \mathrm{d}s$ and X_t diverge a.s.

Set of hypotheses on the interaction potential (H)

- W is $\mathcal{C}^2$, strictly uniformly convex and symmetric,

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- W is $\mathcal{C}^2$, strictly uniformly convex and symmetric,
- there exist $C, k > 0$ such that

$$|W(x)| + |W'(x)| + |W''(x)| \leq C(1 + |x|^k).$$

Results

Theorem

*Suppose that W satisfies the assumption **(H)**. Then there exists a unique probability density function ρ_∞ such that a.s.*

$$\mu_t \to \rho_\infty(\cdot - c_\infty)\mathrm{d}x.$$

Outline

Relation with a Markovian system

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μ_t is asymptotically close to the deterministic dynamical system:
$\dot{\mu} = \Pi(\mu) - \mu$,
where $\Pi(\mu) := \frac{1}{Z(\mu)} e^{-2W*\mu}$.

Strategy of the proof

- Compare on $[T_n, T_{n+1}]$ the trajectories of

$$\mathrm{d}X_t = \mathrm{d}B_t - W' * \mu_t(X_t)\mathrm{d}t$$

with those of the corresponding process where μ_t is replaced by μ_{T_n}:

$$\mathrm{d}Y_t = \mathrm{d}B_t - W' * \mu_{T_n}(Y_t)\mathrm{d}t$$

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- Estimate the speed of convergence of the empirical measure of Y toward the invariant probability measure $\Pi(\mu_{T_n})$

Strategy for the approximation by a dynamical system

- Compare the flow obtained by the “Euler method”

$$\mu_{[T_n,T_{n+1}]} = \mu_{T_n} + \frac{\Delta T_n}{T_{n+1}}\left(\mu_{[T_n,T_{n+1}]} - \mu_{T_n} + \text{error}\right)$$

with the flow

$$\dot{\mu} = \frac{1}{T_n}(\Pi(\mu) - \mu)$$

Outline

Reference point

Definition

The center of the probability measure is the point c_μ such that $W' * \mu(c_\mu) = 0$.
We define the centered measure μ^c as

$$\mu^c(A) = \mu(A + c_\mu).$$

The deterministic system

- Comparison with the Ornstein-Uhlenbeck process

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- Lyapunov function: free energy
- Estimation of the speed of convergence (decrease of the entropy)
- Convergence of the center

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Theorem

*Suppose that W satisfies the hypothese **(H)**. Then:*

1. *there exists a unique probability density function ρ_∞ centered such that a.s.*

$$\mu_t^c \to \rho_\infty(x)\mathrm{d}x,$$

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1. *there exists a unique probability density function ρ_∞ centered such that a.s.*

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2. *a.s. the center $c_t = c(\mu_t)$ converges to a (random) limit c_∞.*