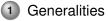
An introduction to self-interacting diffusions

Aline Kurtzmann

Workshop May 2022

05/18/2022







Generalities

2 Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)



Generalities

- 2 Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)
- 3 Discretisation and dynamical system

Generalities

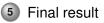
- 2 Self-attracting diffusion on $\mathbb R$ (with Victor Kleptsyn)
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4 Centering

Generalities

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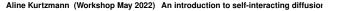
Generalities

Outline



Generalities

Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)



DQ P

Brownian polymer

Durrett and Rogers (1992) on \mathbb{R}^d :

$$\mathrm{d}X_t = \mathrm{d}B_t + \int_0^t f(X_t - X_s) \,\mathrm{d}s \,\mathrm{d}t,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is mesurable and bounded.



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Question: what is the normalisation of *X*? Applications: physics, biology.

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Generalities

Self-attracting case

Studied by:

Cranston & Le Jan (1995): linear and 1 – d constant interaction (f(x) = -a sign(x)),



Generalities

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Self-attracting case

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- Cranston & Le Jan (1995): linear and 1 d constant interaction $(f(x) = -a \operatorname{sign}(x)),$
- Raimond (1997): constant interaction ($d \ge 2$, f(x) = -ax/|x| with a > 0),
- Herrmann & Roynette (2003):

Theorem (Herrmann & Roynette, 2003)

1) Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, decreasing and bounded. Suppose that there exists $C, \rho > 0$ and $k \in \mathbb{N}^*$ such that $|f(x)| \ge Ce^{-\rho/|x|^k}$ around 0. Then X_t converges a.s. 2) When the interaction is not local, $f(x) = -sign(x)\mathbf{1}_{\{|x|\ge a\}}$, then the trajectories remain bounded a.s. (but do not converge).

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Self-repulsive case

Theorem (Mountford & Tarrès, 2007) Let $f(x) = \frac{x}{1+|x|^{1+\beta}}$ with $0 < \beta < 1$. Then, there exists c > 0 such that with probability 1/2, $\frac{X_t}{t^{\alpha}} \rightarrow c$, where $\alpha = \frac{2}{1+\beta}$.



Generalities

Conjecture (Durrett and Rogers)

Theorem (Tarrès & Tóth & Valkó, 2012) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a compact support, $xf(x) \ge 0$ and f(-x) = -f(x). Then $\frac{X_t}{t}$ converges a.s. toward 0.

What is a self-interacting diffusion?

Solution of

$$\mathrm{d}X_t = \mathrm{d}B_t - F(t, X_t, \mu_t)\mathrm{d}t$$

•
$$\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} \mathrm{d}s$$

Reinforced diffusion on a compact set

Benaïm, Ledoux and Raimond (2002), Benaïm and Raimond (2003, 2005) on a compact manifold:

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{t}\int_0^t \nabla_X W(X_t, X_s) \mathrm{d}s \,\mathrm{d}t$$



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Heuristic: show that $\mu_t := \frac{1}{t} \int_0^t \delta_{X_s} ds$ is close to a deterministic flow.

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Difficulty of the study on \mathbb{R}^d

Let

$$\mathrm{d}X_t = \mathrm{d}B_t - (\log t)^3 \nabla W(X_t - \overline{\mu}_t) \mathrm{d}t, X_0 = x$$

where $\overline{\mu}_t = \frac{1}{t} \int_0^t X_s \, \mathrm{d}s$.

Theorem (Chambeu & K)

1 The process $Y_t = X_t - \overline{\mu}_t$ converges a.s. to Y_{∞} , where Y_{∞} belongs to the set of local minima of W. Moreover, for each local minimum m, we have $\mathbb{P}(Y_{\infty} = m) > 0$.



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2 On the set $\{Y_{\infty} = 0\}$, both X_t and $\overline{\mu}_t$ converge a.s. to $\overline{\mu}_{\infty} := \int_0^{\infty} Y_s \frac{ds}{s}$. Moreover, on the set $\{Y_{\infty} \neq 0\}$, we have $\lim_{t \to \infty} X_t / \log t = Y_{\infty}$.

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Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)

Outline

Generalities

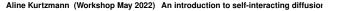


Self-attracting diffusion on $\mathbb R$ (with Victor Kleptsyn)

3 Discretisation and dynamical system

4 Centering





DQ P

Self-attracting diffusion on $\mathbb R$ (with Victor Kleptsyn)

Study

$$dX_t = dB_t - \left(V'(X_t) + \frac{1}{t} \int_0^t W'(X_t - X_s) ds\right) dt$$

= $dB_t - (V'(X_t) + W' * \mu_t(X_t)) dt$
 $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$

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Self-attracting diffusion on R (with Victor Kleptsyn)

Example: quadratic W

Lemma

Let $W(x) = ax^2$ with a > 0. Then a.s. the empirical measure μ_t converges (weakly) to μ_{∞} where $\mu_{\infty}(\cdot - \bar{\mu}_{\infty}) \sim \mathcal{N}(0, 1/a)$ and $\bar{\mu}_{\infty}$ is also a Gaussian variable.



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Let
$$W(x) = \frac{1}{2}(x-1)^2$$
. Then $\overline{\mu}_t = \frac{1}{t} \int_0^t X_s ds$ and X_t diverge a.s.



Set of hypotheses on the interaction potential (H)

• W is C^2 , strictly uniformly convex and symmetric,



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- W is C^2 , strictly uniformly convex and symmetric,
- there exist C, k > 0 such that

$$|W(x)| + |W'(x)| + |W''(x)| \le C(1 + |x|^k).$$



Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)

Results

Theorem

Suppose that W satisfies the assumption (H). Then there exists a unique probability density function ρ_{∞} such that a.s.

 $\mu_t \to \rho_\infty (\cdot - \mathbf{c}_\infty) \mathrm{d}\mathbf{x}.$



Discretisation and dynamical system

Outline

1) Generalities

2) Self-attracting diffusion on $\mathbb R$ (with Victor Kleptsyn)

3 Discretisation and dynamical system

Centering

5 Final result

Discretisation and dynamical system

Relation with a Markovian system



Discretisation and dynamical system

Relation with a Markovian system

 μ_t is asymptotically close to the deterministic dynamical system: $\dot{\mu} = \Pi(\mu) - \mu$,

where $\Pi(\mu) := \frac{1}{Z(\mu)} e^{-2W * \mu}$.



Strategy of the proof

• Compare on $[T_n, T_{n+1}]$ the trajectories of

$$\mathrm{d}X_t = \mathrm{d}B_t - W' * \mu_t(X_t)\mathrm{d}t$$

with those of the corresponding process where μ_t is replaced by μ_{T_n} :

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 Estimate the speed of convergence of the empirical measure of Y toward the invariant probability measure Π(μ_{T_n})

Strategy for the approximation by a dynamical system

Compare the flow obtained by the "Euler method"

$$\mu_{[T_n,T_{n+1}]} = \mu_{T_n} + \frac{\Delta T_n}{T_{n+1}} \left(\mu_{[T_n,T_{n+1}]} - \mu_{T_n} + \operatorname{error} \right)$$

with the flow

$$\dot{\mu} = \frac{1}{T_n}(\Pi(\mu) - \mu)$$

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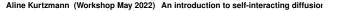
Outline



- 2 Self-attracting diffusion on \mathbb{R} (with Victor Kleptsyn)
- 3 Discretisation and dynamical system







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Reference point

Definition

The center of the probability measure is the point c_{μ} such that $W' * \mu(c_{\mu}) = 0$. We define the centered measure μ^{c} as

$$\mu^{c}(\boldsymbol{A}) = \mu(\boldsymbol{A} + \boldsymbol{c}_{\mu}).$$



The deterministic system

Comparison with the Ornstein-Uhlenbeck process



The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
- Lyapunov function: free energy



The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
- Lyapunov function: free energy
- Estimation of the speed of convergence (decrease of the entropy)

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The deterministic system

- Comparison with the Ornstein-Uhlenbeck process
- Lyapunov function: free energy
- Estimation of the speed of convergence (decrease of the entropy)
- Convergence of the center

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Final result

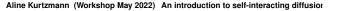
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Suppose that W satisfies the hypothese (H). Then:

1 there exists a unique probability density function ρ_{∞} centered such that a.s.

 $\mu_t^c \to \rho_\infty(\mathbf{x}) \mathrm{d}\mathbf{x},$

2 a.s. the center $c_t = c(\mu_t)$ converges to a (random) limit c_{∞} .

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