A Variational approach to stability and limits of McKean-Vlasov dynamics

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Outline

1. Motivations
2. Statement of the problem
3. Finite temperature limit
4. Low temperature limit

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Equilibrium

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It amounts to describe the system using the thermodynamic formalism, where many objects of interest (e.g. Gibbs measures, equilibrium states) are characterized via thermodynamics functionals (e.g. the free energy, entropy). While the names of the mathematical objects introduced are reminiscent of Statistical Physics, this is now-a-day a rather clean mathematical construction, currently used for most of the state-of-the-art results in Dynamical Systems and geometric dynamics in general.

Macroscopic Fluctuations Theory

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The dynamical counterpart of the (equilibrium) free energy, is provided by large deviations rates of the currents of the system

This approach recovers the classical formalism in the equilibrium case, and provided some experimental (both numerical and in the real world) in genuinely non-equilibrium systems.

MFT for diffusive systems

To fix the ideas, many (diffusive, conservative) particles systems are described at the mesoscopic level as

$$\partial_t \varrho + \mathrm{div}(J_\varrho) = 0$$

$$J_\mu = -D[\mu]\nabla\mu + \chi[\mu] + \frac{1}{\sqrt{\beta N}}\sqrt{a[\mu]}\,\dot{W} + \text{small}$$

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Here D, a are tensor characterizing the diffusion and fluctuation properties of the system, χ is a vector field associated to the response to external fields. There is a precise way to define D, χ, a for a given particles system, provided certain limits exist.

MFT for diffusive systems

The mean-field case

$$\dot{X}_i = b(X_i, \pi_N) + \sqrt{\tfrac{2}{\beta}}\,\dot{W} \qquad \pi_N = \frac{1}{N}\sum_j \delta_{X_j}$$

is readily recovered with the choice $D = 1$, $\chi = \mu\, b(\cdot, \mu)$, $a = \mu$.

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I will discuss exactly this case. The formalism and splitting part of the talk will hold for a general case. But the sharp part about the limits only holds for this case.

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Mathematical framework

We consider a Riemannian manifold M. For $\mu \in \mathcal{P}(M)$ the norm $\|\omega\|_\mu$ stands for the $L^2(\mu)$ norm on 1-forms, and $\|j\|_\mu$ the dual norm on 1-currents.

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We denote $\boldsymbol{\mu}$ a generic curve of probability measure $t \mapsto \mu_t \in \mathcal{P}(M)$, and $\boldsymbol{j}$ a generic curve of 1-currents $t \mapsto j_t \in \mathcal{D}_1(M)$. The space $\mathcal{U}$ denotes the couples $(\boldsymbol{\mu}, \boldsymbol{j})$ such that (weakly)

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$$\partial_t \mu + \operatorname{div}(j) = 0 \tag{1}$$

We fix a tangent field $b \equiv b_\mu(x)$. Then consider J_μ the typical current while in the state μ

$$J_\mu(\omega) := \mu(\langle b_\mu, \omega \rangle) + \mu(d^* \omega), \qquad \omega \in \mathcal{D}^1. \tag{2}$$

Mathematical framework

With this notation the Kolmogorov equation associated to the process interacting with its own law, the limit of the marginal of the mean-field equation, can be just stated as $\boldsymbol{j} = J_{\boldsymbol{\mu}}$. We keep also track of the fluctuations of the original system by rather considering the MFT functional

$$\mathcal{I}(\boldsymbol{\mu}, \boldsymbol{j}) = \frac{\beta}{2} \int_0^T \|j_t - J_{\mu_t}\|_{\mu_t}^2 \tag{3}$$

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Notice that now the nonlinear Kolmogorov equation is just $\mathcal{I}(\boldsymbol{\mu}, \boldsymbol{j}) = 0$. This is nothing but the rate functional for the couple $(\pi_N, \imath_N)$ where $\imath_N$ is the instantaneous empirical current $\imath_N(\omega) = \frac{1}{N} \sum_i \omega(X_{i,t}) \circ dX_{i,t}$.

Stating the problem

We want to understand the stability properties of the minimizers of this functional. In other words, if we move the parameter β and take $b_\mu \equiv b_\mu^\beta$, we want to establish a stability as $\beta \to \beta_0$ or $\beta \to \infty$ (also the case $\beta \to 0$ may feature metastability but we do not investigate it).

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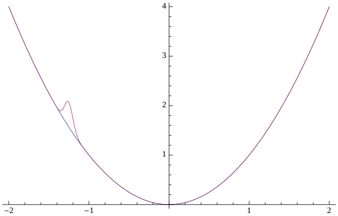
This is not a trivial problem. Even without interaction and for reversible systems, consider

$$\dot{X} = -\nabla V^\beta(X) + \sqrt{\frac{2}{\beta}}\dot{W} \tag{4}$$



How large can we take the little bump with β so that we have convergence to the deterministic dynamics?

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This problem has of course been widely studied in the literature. In particular P.Mathieu obtained that for $V^\beta = V + \beta^{-a} U$, and U regular enough, one has convergence for $a > 1$. There are counterexamples for $a < 1$. We want some improvements on this kind of results:

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- Consider more than 1d.
- Consider non-reversible models.
- Consider non-linear models (interaction with the law).
- Prove convergence of measure-current, not just the law of the process.
- Get some stability for the whole functional, not just the minimizer (keeping track of the fluctuation of the whole system).
- Cover the critical case $a = 1$.

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Let m_μ^β be the invariant measure when the state μ in b_μ^β is frozen. m_μ^β writes as $m_\mu^\beta = \exp(-\beta V_\mu^\beta)\, d\mathrm{vol}$. This is called the local Gibbs measure.

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Theorem

Assume that V_μ^β converges locally uniformly to V_μ (both in x and μ) and that $\partial_\mu V_\mu^\beta$ converges locally uniformly to $\partial_\mu V_\mu^\beta$. Assume that $c_\mu^\beta \to c_\mu$ in C^{-1}.

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Then $\underline{\lim}_\beta\, I^\beta(\boldsymbol{\mu}^\beta, \boldsymbol{j}^\beta) \geq I^{\beta_0}(\boldsymbol{\mu}, \boldsymbol{j})$ whenever $(\boldsymbol{\mu}^\beta, \boldsymbol{j}^\beta) \to (\boldsymbol{\mu}, \boldsymbol{j})$.

Variational formulation

Let us quickly show the proof (of a stronger statement), to get a feeling of the methods involved.

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Define the entropy and the Fisher information, with $h(u) = u \log u$

$$
\begin{aligned}
H(\mu) &= m_\mu(h(\varrho)) \\
I(\mu) &:= \tfrac{1}{2}\|h''(\varrho)d\varrho\|_\mu^2
\end{aligned}
\tag{5}
$$

and define the symmetric and skew-symmetric currents when $H(\mu), I(\mu) < \infty$

$$
S_\mu(\omega) = m_\mu(\langle\, d\varrho, \omega\rangle) \tag{6}
$$

$$
R_\mu(\omega) = \mu\big(\langle c_\mu, \omega\rangle\big) \tag{7}
$$

Variational formulation

Then $J_\mu = S_\mu + R_\mu$ and

$$\mathcal{I}(\boldsymbol{\mu}, \boldsymbol{j}) = H(\mu_T) - H(\mu_0) + \frac{1}{2\beta}\int_0^T I(\mu_t)dt + \frac{\beta}{2}\int_0^T \|j_t - R_{\mu_t}\|^2_{\mu_t} dt \\ - \beta \int_0^T (\mu_t \otimes j_t)\,(\partial_\mu V_\mu)\,dt, \tag{8}$$

where $\partial_\mu V_\mu(x, y)$ is the Lions derivative of V_μ.

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This is an extension of the classical gradient-flow formulation of (reversible) diffusion Kolmogorov equation.
To have minimal regularity assumptions, get rid of R_μ and V_μ.

Variational formulation

Definition

We say that a measurable function $\mathcal{P}(M) \mapsto m_\mu$ taking values in the space of finite Borel measures on M is *log-differentiable* if there exists a measurable map $\mathcal{P}(M) \times M \ni (\mu, x) \mapsto F_\mu(x)$ taking values in the space of measurable 1-forms on M, such that such that for every $(\boldsymbol{\mu}, \boldsymbol{j}) \in \mathcal{M}_0$ and $T > 0$

$$\int_0^T \int_M \|F_{\mu_t}(x)\|^2_{\mu_t} dm_{\mu_t}(x)\, dt < \infty \tag{9}$$

and for each smooth test function $(t, x) \mapsto f_t(x)$

$$m_{\mu_T}(f_T) - m_{\mu_0}(f_0) - \int_0^T m_{\mu_t}(\partial_t f_t)\, dt = -\int_0^T (m_{\mu_t} \otimes j_t)\,(F_{\mu_t}\, f_t) \tag{10}$$

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If $m_\mu = \exp(-V_\mu)$ and V_μ is differentiable, then m_μ is log-differentiable.

Variational formulation

For fixed $\omega \in \mathcal{D}^1$ and μ, consider the (elliptic) equation in the unknown $\Phi \equiv \Phi_\mu^\beta[\omega]$

$$m_\mu^\beta(\langle \nabla \Phi, df \rangle) = m_\mu(f \langle c_\mu^\beta, \omega \rangle) - m_\mu^\beta(\langle c_\mu^\beta, \omega \rangle) m_\mu^\beta(f) \qquad f \in C_c^\infty(M) \tag{11}$$

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Remark

R_μ writes in terms of $\Phi_\mu[\cdot]$. However, $R_\mu(\omega)$ is not determined by $\Phi_\mu[\omega]$ for a fixed ω.

Sketch of the proof

In the variational formula, write R_μ in terms of Φ^β_μ, and $F^\beta_\mu = \partial_\mu \log m^\beta_\mu$ in place of $\partial_\mu V_\mu$. Then pass to the limit in every term in (8).

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In other words

$$b_\mu \qquad \leftrightarrow V_\mu, c_\mu \qquad \leftrightarrow m_\mu, \Phi_\mu$$

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Main Theorem

The same strategy is used in the limit $\beta \to \infty$. However in this case, each term in formula (8) is generate.

Theorem

Assume $V_\mu^\beta = \bar{V}_\mu^\beta + \frac{1}{\beta} U_\mu^\beta$, $\bar{V}_\mu^\beta \to V_\mu$ *in* C^1 *(uniformly in* μ*) and* U^β *precompact in* C^0*.*

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Assume that $c_\mu^\beta = \bar{c}_\mu^\beta + \frac{1}{\beta} r_\mu^\beta$ *with* $c_\mu^\beta \to c_\mu$ *un* C^0 *and* $r_\mu^\beta \to 0$ *in* C^{-1}*.*
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Assume moreover $\nabla V_\mu \cdot c_\mu = 0$
Assume $\partial_\mu V_\mu^\beta \to \partial_\mu V_\mu$ locally uniformly.

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Assume moreover $\nabla V_\mu \cdot c_\mu = 0$
Assume $\partial_\mu V^\beta_\mu \to \partial_\mu V_\mu$ *locally uniformly.*
Assume suitable conditions on the initial condition and at infinity.
Then whenever $(\boldsymbol{\mu}^\beta, \boldsymbol{j}^\beta) \to (\boldsymbol{\mu}, \boldsymbol{j})$

$$\varliminf_{\beta} \frac{1}{\beta} \mathcal{I}^\beta(\boldsymbol{\mu}^\beta, \boldsymbol{j}^\beta) \geq \frac{1}{2} \int_0^T \| j_t - \mu_t(-\nabla V_{\mu_t} + c_{\mu_t}) \|^2_{\mu_t} \, dt$$

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Notice that this implies that X^β (and its current) converges in probability to solutions (uniqueness of the limit is not needed/proved)

$$\dot{X} = -\nabla V_\mu(X) + c_\mu(X) \tag{12}$$

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Notice also that if $\beta^{1-\alpha}\|V_\mu^\beta - V_\mu\|_{W^{\alpha,\infty}} \to 0$, then the hypotheses are satisfied (this recovers some of the literature).

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- If β stays bounded, this immediately yields convergence results for the functional $\mathcal{I}^\beta$ (and thus for the random dynamics).
- If $\beta \to \infty$, the decomposition of $\mathcal{I}^\beta$ becomes generate in the limit, but gives sharp results of convergence to the deterministic limit.