Large deviations for singularly interacting diffusions

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McKean-Vlasov SDE on $\mathbb{R}^d$:

$$d\bar{X} = b(\bar{X}, \text{Law}(\bar{X}))dt + dW$$

Associated interacting particle system:

$$dX^i = b(X^i, \frac{1}{N} \sum_{j=1}^{N} \delta_{X^i} \delta_{X^j})dt + dW^i, \quad i = 1, \ldots, N$$

**Singular** interaction drift $b$, e.g. ($d \geq 2$):

$$b(x, \mu) = \int \varphi(x - y) \mu(dy), \quad \varphi(x) \approx |x|^\alpha, \quad \alpha > -1.$$ 

We will show, in the limit $N \to \infty$:

- **large deviation principle** for interacting particle system
- **convergence** to McKean-Vlasov SDE
McKean-Vlasov SDEs: SDEs where the drift depends on the law of the solution:

\[ d\bar{X} = b(\bar{X}, \text{Law}(\bar{X})) dt + dW \]

Associated mean-field interacting particle system:

\[ dX^{i,N} = b(X^{i,N}, \frac{1}{N} \sum_{j=1}^{N} \delta_{X^{j,N}}) dt + dW^i, \quad i = 1, \ldots N \]

**Theorem (e.g. Sznitman 91)**

If \( b \) is Lipschitz and bounded in the 1-Wasserstein topology, then

- existence and uniqueness of MV SDE
- convergence of interacting particle system to MV SDE

\[ \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}} \rightarrow \text{Law}(\bar{X}) \text{ as } N \rightarrow \infty \]
As a consequence, if \( \mu \neq \text{Law}(\bar{X}) \), then \( P\left( \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{i,N}} \approx \mu \right) \to 0 \) (as \( N \to \infty \)). Question: how fast? large deviations:

**Theorem (e.g. Budhiraja-Dupuis-Fischer 12)**

If \( b \) is Lipschitz and bounded, then LDP (on \( \mathcal{P}(C([0, T])^d) \)) with (speed \( N \) and) rate function \( R(\mu \mid \text{Law}(X^\mu)) \)

\[
P \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{i,N}} \approx \mu \right) \approx \exp[-NR(\mu \mid \text{Law}(X^\mu))] 
\]

where \( R \) is the relative entropy and \( X^\mu \) is defined below:

\[
R(\mu \mid \nu) = \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu \\
dX^\mu = b(X^\mu, \mu)dt + dW
\]
Question: what happens for non smooth $b$?

Two cases:

- **Gibbs 2-point interaction:**

  $$b(x, \mu) = \int \varphi(x - y) \mu(dy)$$

  Typically $\varphi$ singular at $x - y = 0$ (problems when particles meet or come close)

- **non-Gibbs interaction:**

  $$b(x, \mu) = \psi \left( x, \mu, \int \varphi(x - y) \mu(dy) \right)$$

  $\psi$ smooth, typically $\varphi$ singular at $x - y = 0$
Types of singularity (for interaction kernel $\varphi$):

- **Subcritical**: on small scale, drift is much smaller than diffusion

  $$\varphi(x) \approx |x|^\alpha \text{ with } \alpha > -1$$

- **Critical**: on small scale, drift is comparable to diffusion: in 2D (Gibbs case):

  $$\varphi(x) = -\frac{x}{|x|^2} \quad \text{2D Keller-Segel}$$

  $$\varphi(x) = \frac{x_\perp}{|x|^2} \quad \text{2D Navier-Stokes}$$
Interacting particle system:

\[ dX^{i,N} = b(X^{i,N}, \frac{1}{N} \sum_{j=1}^{N} \delta_{X^{j,N}})dt + dW^i, \quad i = 1, \ldots, N \]

where

\[ b(x, \mu) = \int \varphi(x - y) \mu(dy) \quad \text{(Gibbs case)} \]

\[ b(x, \mu) = \psi \left( x, \mu, \int \varphi(x - y) \mu(dy) \right) \quad \text{(non-Gibbs case)} \]
Theorem ((part 1))

Assume (subcritical case) that \( \varphi \in L_t^q(L_x^p) \) with

\[
\frac{d}{p} + \frac{2}{q} < 1, \quad p, q \geq 2
\]

Then an LDP holds for \( \frac{1}{N} \sum_{i=1}^N \delta_{X_i,N} \in \mathcal{P}(C([0, T])^d) \) with weak topology, with rate function \( R(\mu \mid \text{Law}(X^\mu)) \):

\[
P \left( \frac{1}{N} \sum_{i=1}^N \delta_{X_i,N} \approx \mu \right) \approx \exp[-NR(\mu \mid \text{Law}(X^\mu))]
\]

with

\[
dX^\mu = b(X^\mu, \mu)dt + dW
\]
Theorem ((part 2))

Moreover, \( \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{iN}} \) converges (in probability) to the unique Law(\(\bar{X}\)) solution to MV SDE.

This includes a large class of subcritical drifts:

- Gibbs and non-Gibbs cases
- \( \varphi(x) \approx |x|^\alpha \) for \( \alpha > -1 \) (\( \alpha > -1/2 \) for \( d = 1 \))
Some known results on convergence (no claim of completeness):

- Jabin-Wang 18: Gibbs subcritical and critical case
  \( \varphi \in W^{-1,\infty} \) with \( \text{div} \varphi \in W^{-1,\infty} \)

- Godinho-Quininao 15: Gibbs subcritical case \( \varphi = -\nabla |x|^{\alpha+1}, \quad \alpha > -1 \)

- Tomasevic 20: Gibbs subcritical case \( \varphi \in L_t^q(L_x^p), \quad d/p + 2/q < 1 \) (+ a continuity condition)

- Fournier-Jourdain 17: 2D Keller-Segel

- Fournier-Hauray-Mischler 14: 2D Navier-Stokes

- Lacker 18: non-Gibbs subcritical case, \( b \) bounded and continuous in the measure argument

- Jabir 19: non-Gibbs subcritical case, assuming weak uniqueness + an integrability condition

- Hao-Röckner-Zhang 22: non-Gibbs subcritical case, general
Some known results on large deviations (no claim of completeness):

- Dawson-Gärtnner 87, DelMoral-Zajic 03, Budhiraja-Dupuis-Fischer 12: LDP assuming continuous drift;
- Lacker 18: subcritical case large deviation upper bound, $b$ bounded and continuous in the measure argument
- Fontbona 04: critical Gibbs 1D case, $\varphi(x) = 1/x$ (repulsive).
Notation:

\[ L^{W,N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{W_j} \text{ (independent (Brownian) particles)} \]

\[ L^{X,N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_{j,N}} \text{ (dependent particles)} \]
Formal idea of the proof (from at least DaiPra-denHollander 96):

1. **Independent case:** Sanov theorem: LDP for $L^{W,N}$ ($W_i$ i.i.d.)
   
   $$P(L^{W,N} \approx \mu) \approx e^{-NI(\mu)}$$

   with $I(\mu) = R(\mu \mid \text{Law}(W^1))$

2. **Density for the dependent case:** Girsanov theorem
   
   $$\text{Law}(X^1, N, \ldots, X^N, N) = e^{-N\mathcal{E}(L^{W,N})} \text{Law}(W^1, \ldots, W^N)$$

3. **Mean field interaction:**
   
   $$\text{Law}(X^1, N, \ldots, X^N, N) = e^{-N\mathcal{E}(L^{W,N})} \text{Law}(W^1, \ldots, W^N)$$

   for a suitable energy $\mathcal{E}$; as a consequence
   
   $$\text{Law}(L^{X,N}) = e^{-N\mathcal{E}(L^{W,N})} \text{Law}(L^{W,N})$$

4. **LDP for change of measure:** Varadhan lemma:
   
   $$P(L^{X,N} \approx \mu) \approx e^{-N\mathcal{E}(\mu)} P(L^{W,N} \approx \mu) \approx e^{-N(\mathcal{E}+I)(\mu)}$$

Formally $\mathcal{E}(\mu) + I(\mu) = R(\mu \mid \text{Law}(X^\mu))$. 

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The energy $\mathcal{E}$:

$$
\mathcal{E}(L^{W,N}) = \frac{1}{N} \sum_{i=1}^{N} \left[ - \int_0^T b(W^i, L^{W,N}) dW^i + \frac{1}{2} \int_0^T |b(W^i, L^{W,N})|^2 dt \right]
$$

$$
= \int \left[ - \int_0^T b(\gamma, L^{W,N}) d\gamma + \frac{1}{2} \int_0^T |b(\gamma, L^{W,N})|^2 dt \right] L^{W,N}(d\gamma)
$$

Gibbs case ($b(x, \mu) = \int \varphi(x - y) d\mu$):

$$
\mathcal{E}(\mu) = \int \int \int V(\gamma^1, \gamma^2, \gamma^3) d\mu \otimes^3 (\gamma^1, \gamma^2, \gamma^3),
$$

$$
V(\gamma^1, \gamma^2, \gamma^3) = - \int_0^T \varphi(\gamma^1_t - \gamma^2_t) d\gamma^1_t + \frac{1}{2} \int_0^T \varphi(\gamma^1_t - \gamma^2_t) \cdot \varphi(\gamma^2_t - \gamma^3_t) dt
$$
Rigorously:

- Varadhan lemma asks the energy $\mathcal{E}$ to be continuous: problem when $\varphi$ is singular
- the energy $\mathcal{E}$ involves a stochastic integral, which is a non-smooth function of the path: $\mathcal{E}$ is not continuous even for smooth drifts (but an LDP is known for smooth $b$)
- Girsanov theorem requires suitable sufficient conditions (e.g. Novikov)

Idea: proceed by approximation:

1. take $\varphi_\lambda$ smooth bounded approximations of the interaction kernel $\varphi$;
2. LDP with interaction kernel $\varphi_\lambda$ holds by classical result;
3. let $\lambda \to 0$ and try to get an LDP with $\varphi$. 

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Large deviations for singularly interacting diffusions
Notation ($\varphi$ interaction kernel):

- $b$ drift
  
  \[
  b(x, \mu) = \int \varphi(x - y)\mu(dy) \text{ (Gibbs case)}
  \]
  
  \[
  b(x, \mu) = \psi \left( x, \mu, \int \varphi(x - y)\mu(dy) \right) \text{ (non-Gibbs case)}
  \]

- $\mathcal{E}$ energy (log-density in Girsanov):
  
  \[
  \mathcal{E}(L^{W,N}) = \int \left[ - \int_0^T b(\gamma, L^{W,N})d\gamma + \frac{1}{2} \int_0^T |b(\gamma, L^{W,N})|^2 dt \right] L^{W,N}
  \]

- In the Gibbs case, $V$ potential:
  
  \[
  \mathcal{E}(\mu) = \int \int \int V(\gamma^1, \gamma^2, \gamma^3) d\mu \otimes^3 (\gamma^1, \gamma^2, \gamma^3),
  \]
  
  \[
  V(\gamma^1, \gamma^2, \gamma^3) = - \int_0^T \varphi(\gamma^1 - \gamma^2)d\gamma^1 + \frac{1}{2} \int_0^T \varphi(\gamma^1 - \gamma^2) \cdot \varphi(\gamma^1 - \gamma^3) dt
  \]

$b$, $\mathcal{E}$, $V$ defined similarly with $\varphi$. 

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Three main points:

1. **Extended Varadhan lemma**: For general change of measures, find sufficient conditions on $\mathcal{E}_\lambda$ to infer the LDP for $\mathcal{E} \to \text{exponentially good approximation } \mathcal{E}_\lambda(L^W,N)$.

2. **Gibbs measures with singular potential**: In the context of Gibbs energy

   \[
   \mathcal{E}(\mu) = \int V(\gamma_1, \ldots, \gamma^k) d\mu^\otimes k(\gamma_1, \ldots, \gamma^k)
   \]

   find sufficient conditions of $V$ and its approximation $V_\lambda$ to meet the exponentially good approximation for $\mathcal{E}_\lambda(L^W,N)$.

3. **Gibbs-like case**: In the context of non-Gibbs energy, reduce the exponentially good approx property (condition on $N$ particles) to a condition on $k$ particles.

4. **MV SDEs with singular drift**: In the context of MV SDE, find sufficient conditions on $\varphi$ and its approximation $\varphi_\lambda$ to meet the conditions of Gibbs or Gibbs-like case.
**Extended Varadhan lemma**: change of measure with discontinuous log-densities:
Here \( L^W, N \) is a generic random variable on a Polish space \( S \), \( \mathcal{E} \) and \( \mathcal{E}_\lambda \) are functions on \( S \).

**Theorem**

**Assume that**

- \( L^W, N \) satisfies an LDP
- for \( \lambda > 0 \), \( \mathcal{E}_\lambda \) induces an LDP (i.e. \( e^{-N\mathcal{E}_\lambda} \text{Law}(L^W, N) \) satisfies an LDP)
- exponentially good approximation: for every \( \beta > 0 \),

\[
\limsup_{N} \frac{1}{N} \log E[e^{N\beta|\mathcal{E}_\lambda|(L^W, N)}] \to 0 \text{ as } \lambda \to 0
\]

Then \( \mathcal{E} \) induces an LDP (i.e. \( e^{-N\mathcal{E}} \text{Law}(L^W, N) \) satisfies an LDP).
**Remark:** we do not require $\mathcal{E}_\lambda$ to be continuous, but only to induce an LDP.

The proof is based on convex analysis tools.

Other known results (no claim of completeness):

- Dembo-Zeitouni 10: exponential approximation
- Eichelsbacher-Schmock 02, Liu-Wu 20: in context of singular Gibbs measure, LDP in a stronger topology (by exponential approximation)
- DelMoral-Zajic 03: another extension of Varadhan lemma
In Gibbs case, exponentially good approximation means

$$\limsup_{N} \frac{1}{N} \log E \left[ \exp \left( N \beta \frac{1}{N^3} \sum_{1 \leq i,j,k \leq N} |V - V_{\lambda}|(W^i, W^j, W^k) \right) \right] \to 0$$

Question: which conditions on $V$ guarantee the exponentially good approximation?
LDP for Gibbs measures with singular potential $V$

Here $W^1, \ldots, W^N$ are i.i.d. random variables on a Polish space $\chi$ and

$$L_{W,N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{W_i}$$

are random variables on $\mathcal{P}(\chi)$ endowed with weak topology.

**Sanov theorem:** $L_{W,N}$ satisfy an LDP with rate function $I(\mu) = R(\mu \mid \text{Law}(W^1))$. 
Interaction:

\[ \mathcal{E}(\mu) = \int V(\gamma^1, \gamma^2, \gamma^3) d\mu^\otimes 3 \]

\( V \) possibly singular.

\( V_\lambda \) approximations of \( V \), \( \mathcal{E}_\lambda \) defined as \( \mathcal{E} \) but replacing \( V \) with \( V_\lambda \).

**Proposition**

Assume that

- for \( \lambda > 0 \), \( \mathcal{E}_\lambda \) induces an LDP (i.e. \( e^{-N\mathcal{E}_\lambda} \text{Law}(L^W,\bar{N}) \) satisfies an LDP)
- for every \( \beta > 0 \),

\[ \log E[\exp[\int_S \beta |V - V_\lambda|(W^1, W^2, W^3)]] \to 0 \text{ as } \lambda \to 0 \]

Then \( \mathcal{E} \) induces an LDP (i.e. \( e^{-N\mathcal{E}} \text{Law}(L^W,\bar{N}) \) satisfies an LDP).
Recall exponentially good approximation (here two point interaction for simplicity):

$$\limsup_N \frac{1}{N} \log E[\exp[N\beta \frac{1}{N^2} \sum_{1\leq i,j\leq N} |V - V_\lambda|(W^i, W^j)]] \to 0$$

**Idea of the proof:** Hoeffding decomposition: rearrange the terms $|V - V_\lambda|(W^i, W^j)$ to get averages of $\approx N$ independent terms:

$$\frac{1}{N^2} \sum_{1\leq i,j\leq N} f(W^i, W^j) \approx \frac{1}{N!} \sum_{\sigma \in S_N} \frac{1}{[N/2]} \sum_{i=1}^{[N/2]} f(W^{\sigma(2i-1)}, W^{\sigma(2i)})$$
\[ \frac{1}{N^2} \sum_{1 \leq i,j \leq N} f(W^i, W^j) \approx \frac{1}{N!} \sum_{\sigma \in S_N} \frac{[N/2]}{1} \sum_{i=1}^{[N/2]} f(W^{\sigma(2i-1)}, W^{\sigma(2i)}) \]

Then, first by Jensen inequality then by independence,

\[ E[\exp[\frac{1}{N^2} \sum_{1 \leq i,j \leq N} |V - V_\lambda|(W^i, W^j)]] \]

\[ \leq C \frac{1}{N!} \sum_{\sigma \in S_N} E[\exp[\frac{1}{N} \sum_{i=1}^{[N/2]} |V - V_\lambda|(W^{\sigma(2i-1)}, W^{\sigma(2i)})]] \]

\[ \leq C \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{i=1}^{[N/2]} E[\exp[c \beta |V - V_\lambda|(W^{\sigma(2i-1)}, W^{\sigma(2i)})]] \]

\[ = CE[\exp[c \beta |V - V_\lambda|(W^1, W^2)]]^{[N/2]} \]
Other known results (no claim of completeness):

- Eichelsbacher-Schmock 02: LDP in a stronger topology, where $\mathcal{E}$ is continuous
- Liu-Wu 20: LDP in a stronger topology, result very close to ours (even more general)
- Berman 18: critical singularity when state space is $\mathbb{R}^d$ (+ other assumptions)
**Gibbs-like framework:** $\mathcal{E}$ no more of the form $\int Vd\mu^{\otimes k}$, but ...

**Proposition**

Assume that

- for $\lambda > 0$, $\mathcal{E}_\lambda$ induces an LDP (i.e. $e^{-N\mathcal{E}_\lambda} \text{Law}(L^W,N)$ satisfies an LDP)
- there exist $V, V_\lambda$ such that, for every $\beta > 0$,
  \[
  \log E[e^{N\beta |\mathcal{E} - \mathcal{E}_\lambda|(L^W,N)}] \leq C \log E[e^{N\frac{c\beta}{N^3} \sum_{1 \leq i,j,k \leq N} |V - V_\lambda|(W^i,W^j,W^k)}]
  \]
- for every $\beta > 0$,
  \[
  \log E[\exp[\int_S \beta |V - V_\lambda|(W^1,W^2,W^3)]] \to 0 \text{ as } \lambda \to 0
  \]

Then $\mathcal{E}$ induces an LDP (i.e. $e^{-N\mathcal{E}} \text{Law}(L^W,N)$ satisfies an LDP).
In MV SDEs, Gibbs case \((b(x, \mu) = \int \varphi(x - y) \mu(\,dy))\),

\[
V(W^1, W^2, W^3) = -\int_0^T \varphi(W^1, W^2) \,dW^1 + \frac{1}{2} \int_0^T \varphi(W^1 - W^2) \cdot \varphi(W^1 - W^3) \,dt
\]

Question: which condition on \(\varphi\) meet the assumption (below) of the Gibbs case?

\[
\log \mathbb{E}[\exp[\int_S \beta |V - V_\lambda|(W^1, W^2, W^3)]] \to 0
\]

In MV SDEs, non-Gibbs case:

\[
\mathcal{E}(L^{W,N}) = \int \left[ -\int_0^T b(\gamma, L^{W,N}) \,d\gamma + \frac{1}{2} \int_0^T |b(\gamma, L^{W,N})|^2 \,dt \right] L^{W,N}(d\gamma)
\]

Question: can one show a Gibbs-like structure? bound on

\[
\frac{1}{N} \log \mathbb{E}[e^{-N\beta|\mathcal{E} - \mathcal{E}_\lambda|(L^{W,N})}]
\]

In all cases we must control stochastic exponentials.
LDP for MV SDEs with singular drift

Interacting particle system:

\[ dX^{i,N}_t = b(X^{i,N}_t, \frac{1}{N} \sum_{j=1}^{N} \delta_{X^{j,N}_t}) \, dt + dW^i_t, \quad i = 1, \ldots, N \]

where

\[ b(x, \mu) = \int \varphi(x - y) \mu(dy) \quad \text{(Gibbs case)} \]

\[ b(x, \mu) = \psi \left(x, \mu, \int \varphi(x - y) \mu(dy)\right) \quad \text{(non-Gibbs case)} \]
Theorem

Assume that, for every $\beta > 0$,

$$\log E \left[ \exp \left( \beta \int_0^T |\varphi - \varphi_\lambda|^2(W^1, W^2)dt \right) \right] \to 0 \text{ as } \lambda \to 0$$

Then $L^{X,N}$ satisfies an LDP with rate function $R(\mu \mid \text{Law}(X^\mu))$. 

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Large deviations for singularly interacting diffusions
Idea of the proof, Gibbs case $(b(x, \mu) = \int \varphi(x - y) d\mu)$:

Recall:

$$V(W^1, W^2, W^3) = -\int_0^T \varphi(W^1 - W^2) dW^1 + \frac{1}{2} \int_0^T \varphi(W^1 - W^2) \cdot \varphi(W^1 - W^3) dt$$

Use exponential martingale bounds to show that

$$\log E[\exp[\int_0^T \beta |V - V_\lambda|(W^1, W^2, W^3)]] \leq C \log E[\exp[c_\beta \int_0^T |\varphi - \varphi_\lambda|^2(W^1, W^2) dt]]$$

Thanks to the assumption on the RHS, apply the criterion for Gibbs measures with singular potential.
Idea of the proof, non-Gibbs (general) case:

Recall:

$$\mathcal{E}(L^W,^N) = \int \left[ - \int_0^T b(\gamma, L^W,^N) \, d\gamma + \frac{1}{2} \int_0^T |b(\gamma, L^W,^N)|^2 \, dt \right] L^W,^N(d\gamma)$$

Also here use exponential martingale bounds to show that

$$\frac{1}{N} \log E\left[e^{-N\beta |\mathcal{E} - \mathcal{E}_\lambda|(L^W,^N)}\right]$$

$$\leq C \frac{1}{N} \log E\left[\exp\left[-Nc_\beta \frac{1}{N^2} \sum_{i,j=1}^{N} \int_0^T |\varphi - \varphi_\lambda|^2(W^i, W^j) \, dt\right]\right]$$

and apply the Gibbs-like framework.
Proposition

Sufficient condition to guarantee assumption on $\varphi$: $\varphi \in L_t^q(L_x^p)$ with $d/p + 2/q < 1$.

The proof is based on:

- Khasminskii Lemma

$$\sup_{x, y \in \mathbb{R}^d} E\left[ \int_0^T |\varphi|^2(x + W^1, y + W^2) dt \right] < \infty$$

$$\Rightarrow E\left[ \exp \int_0^T |\varphi|^2(W^1, W^2) dt \right] < \infty$$

- heat kernel bounds to show that

$$E\left[ \int_0^T |\varphi|^2(x + W^1, y + W^2) dt \right] < \infty$$
Uniqueness for the McKean-Vlasov SDE: idea of the proof in the Gibbs case:

Call $\mathcal{F} : \mathcal{P}(C([0, T]^d)) \ni \mu \mapsto \text{Law}(X^\mu) \in \mathcal{P}(C([0, T]^d))$, where

$$dX^\mu = b(X^\mu, \text{Law}(X^\mu))dt + dW$$

Show that $\mathcal{F}$ is a contraction (for small $T$) on

$$\{ \mu \mid \left\| \frac{d\mu}{d\text{Law}(W)} \right\|_{L^m} < \infty \}.$$
\[
E^1 \left[ \left| \frac{d\mu}{d\text{Law}(W)} - \frac{d\nu}{d\text{Law}(W)} \right|^m (W^1) \right] \\
\leq E^1 \left[ \left| \int \int_{0}^{T} \varphi(W^1, y) dt (\mu - \nu)(dy) \right|^{2m+1/m+} \right] \\
\leq E^1 \left[ E^2 \left[ \int_{0}^{T} \varphi(W^1, W^2) dt \left( \frac{d\mu}{d\text{Law}(W)} - \frac{d\nu}{d\text{Law}(W)} \right)(W^2) \right]^{2m+} \right] \\
\leq E \left[ \left( \int_{0}^{T} |\varphi(W^1, W^2)|^2 dt \right)^{m+1/m+} \right] \\
\cdot E^2 \left[ \left| \frac{d\mu}{d\text{Law}(W)} - \frac{d\nu}{d\text{Law}(W)} \right|^2 (W^2) \right]^m 
\]
Results:

- LDP and convergence in Gibbs and non-Gibbs cases with singular interaction kernel in the subcritical class (by Itô calculus and Khasminskii lemma)
- extended Varadhan lemma (by convex integration tools)
- LDP for singular Gibbs measures (by Hoeffding decomposition)
Possible perspectives:

- LDP for some drifts in the critical class?
- other approaches (suited e.g. for degenerate noise)?
- application to metastability?
Thank you!