Dynamic behavior of meanfield systems: Phase transitions, metastability, and self-similarity

Metastability, mean-field particle systems and non-linear processes

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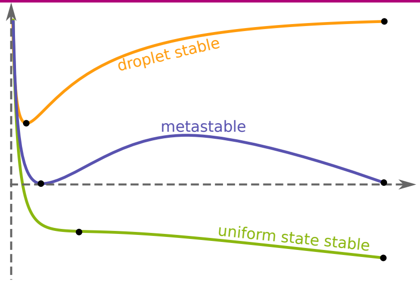
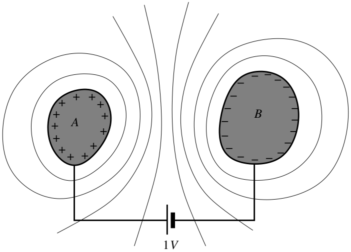
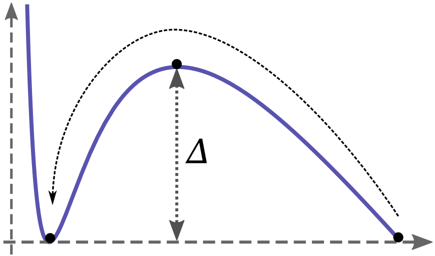


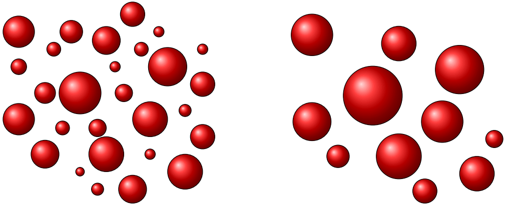
Mean-field systems and phase transitions



Metastability and potential theory



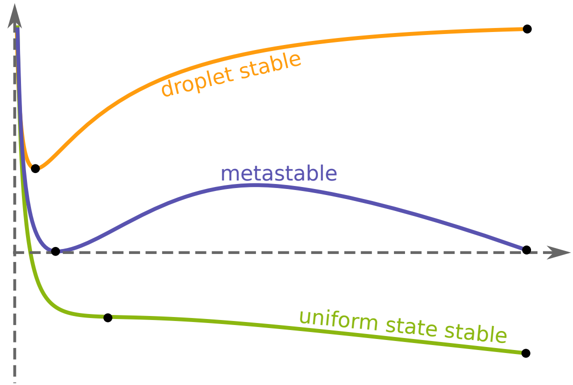
Rare events and mountain-passes



Phase separation in cluster growth

Phase transitions in the McKean–Vlasov model

[Carrillo-Gvalani-Paviliotis-S. '20]



The McKean–Vlasov equation – Derivation

- Overdamped Langevin equation defined on $\mathbb{T}_L^d \simeq [0,L)^d$

$$\mathrm{d}X_t^i = -\frac{\kappa}{N}\sum_{j=1, j\neq i}^{N} \nabla W(X^i - X^j)dt + \sqrt{2}\,\mathrm{d}B_t^i \qquad , i = 1,\dots,N$$

- $\kappa \in [0,\infty)$ interaction strength (bifurcation parameter)
- The mean-field limit $N \to \infty$ is governed by the McKean–Vlasov equation

$$\partial_t \varrho = \Delta\varrho + \kappa\nabla\cdot(\varrho\nabla W \star \varrho) \qquad \text{in } \mathbb{T}_L^d \times (0,T]$$

- properties encoded in interaction potential $W : \mathbb{T}_L^d \to \mathbb{R}$ (coordinate-wise even)

Some applications: Models for finite N or mean-field limit include

- Molecular dynamic (Lennard–Jones, Van-der-Waals)
- Collective motion of agents (attractive-repulsive)
- Opinions of individuals (Hegselmann–Krause)
- Liquid crystals / nanorods (anisotropic, Onsager, Maier–Saupe)
- Nonlinear synchronizing oscillators (Kuramoto)
- Chemotaxis models (Patlak–Keller–Segel)

Example: Nonlinear synchronization of oscillators

The Kuramoto model: $W(x) = -\cos x$ and $L = 2\pi$

$\kappa < \kappa_c$, no phase locking

$\kappa > \kappa_c$, phase locking

Example: 2d Gaussian attractive interaction potential

$$W(x) = -\frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}$$

with $\sigma^2 = \frac{1}{2}$, $L = 10$, $\kappa = \sqrt{2L} > \kappa_c$.

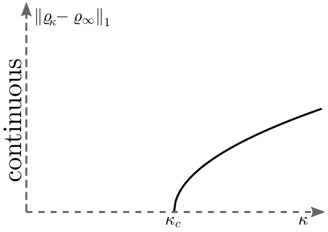
Transition points and types of phase transitions

Free energy functional (Lyapunov property, gradient flow)

$$\mathcal{F}_\kappa(\varrho) = \int_{\mathbb{T}_L^d} \varrho \log \varrho \,\mathrm{d}x + \frac{\kappa}{2} \iint_{\mathbb{T}_L^d \times \mathbb{T}_L^d} W(x-y)\varrho(x)\varrho(y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

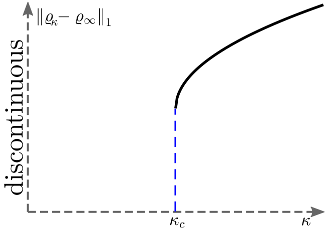
Definition: Let $\varrho_\infty \equiv L^{-d}$. κ_c is transition point, if:

- For $\kappa \le \kappa_c$ is ϱ_∞ global minimizer of $\mathcal{F}_\kappa$ and unique for $\kappa < \kappa_c$
- For $\kappa > \kappa_c$ exists another global minimizer ϱ_κ



Results and Goals:

- Bifurcation analysis and local stability around $\varrho_\infty \equiv L^{-d}$
- Classification for continuous and discontinuous transitions
- Understanding of the free energy landscape
- Dynamical properties related to nucleation and coarsening



Characterization of phase transition

Theorem [Carrillo-Gvalani-Paviliotis-S. '20]

Let $\widetilde{W} : \mathbb{N}^d \to \mathbb{R}$ denote the (real) Fourier modes of W.

- If there is only one dominant unstable mode k^*: For $\alpha > 0$ small enough holds

$$\alpha \widetilde{W}(k^*) \leq \widetilde{W}(k) \qquad \text{for all } k \neq k^* : \widetilde{W}(k) < 0 \ ,$$

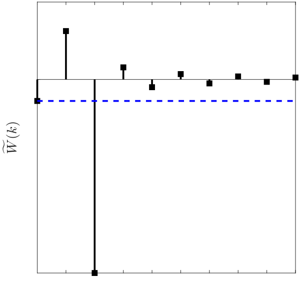
  then the transition point κ_c is continuous.

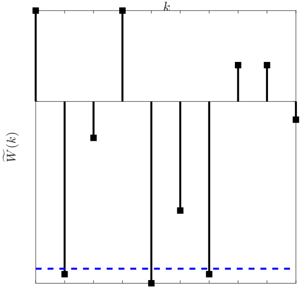
- If there exist (near)-dominant resonating modes k^a, k^b, k^c: That is for δ small enough exist

$$k^a, k^b, k^c \in \left\{ k' \in \mathbb{N}^d : \widetilde{W}(k') \leq \min_{k \in \mathbb{N}^d} \widetilde{W}(k) + \delta \right\} \quad \text{with } k^a = k^b + k^c,$$

  then the transition point κ_c is discontinuous.

$\Rightarrow$ local attractive potentials lead to discontinuous phase transitions





Basic properties of transition points

Summary of critical points:

- κ_c transition point
- κ_* bifurcation point
- $\kappa_\sharp$ point of linear stability, i.e., $\kappa_\sharp = -\frac{L^{\frac{d}{2}}}{\beta \min_k \tilde{W}(k)/\Theta(k)}$.
  If $k_\sharp = \arg\min \tilde{W}(k)$ is unique, then $\kappa_\sharp = \kappa_*$ is a bifurcation point.

Results from [Gates & Penrose 1970] and [Chayes & Panferov '10]

- $\mathscr{F}_\kappa$ has a transition point κ_c iff $W \notin \mathbb{H}_s$
- $\min \mathscr{F}_\kappa$ is non-increasing as a function of κ
- If for some κ' : ϱ_∞ is no longer the unique minimiser,
  then $\forall \kappa > \kappa'$: ϱ_∞ is no longer a unique minimizer
- If κ_c is continuous, then $\kappa_c = \kappa_\sharp$

Conclusion:

- To proof a discontinuous transition: Show ϱ_∞ is no longer global minimizer at $\kappa_\sharp$.
- To proof a continuous transition:
  If $\kappa_* = \kappa_\sharp$, sufficient to show that ϱ_∞ at $\kappa_\sharp$ is the unqiue global minimizer.

Argument for resonating dominant modes ($\delta = 0$)

Let $\varepsilon > 0$ be sufficiently small such that $\varrho = \varrho_\infty\left(1 + \varepsilon \sum_{k \in K^\delta} w_k\right) \in \mathcal{P}^+(\mathbb{T}^d)$.

Entropy and energy of Ansatz:

$$\beta^{-1} S(\varrho) = \beta^{-1}\left(S(\varrho_\infty) + \frac{|K^\delta|}{2}\varrho_\infty \varepsilon^2 - \frac{\varrho_\infty}{3}\int \varepsilon^3 \Big(\sum_{k \in K^\delta} w_k\Big)^3 + O(\varepsilon^4)\right)$$

$$\frac{\kappa_\sharp}{2}\mathcal{E}(\varrho, \varrho) = \frac{\kappa_\sharp}{2}\mathcal{E}(\varrho_\infty, \varrho_\infty) + \frac{\kappa_\sharp \varepsilon^2 |K^\delta| \varrho_\infty^2}{2} \min_{k \in \mathbb{N}^d} \frac{\tilde{W}(k)}{\Theta(k)} L^{d/2}$$

Combining both estimates, recalling $\kappa_\sharp = -\frac{L^{\frac{d}{2}}}{\beta \min_k \tilde{W}(k)/\Theta(k)}$, yields

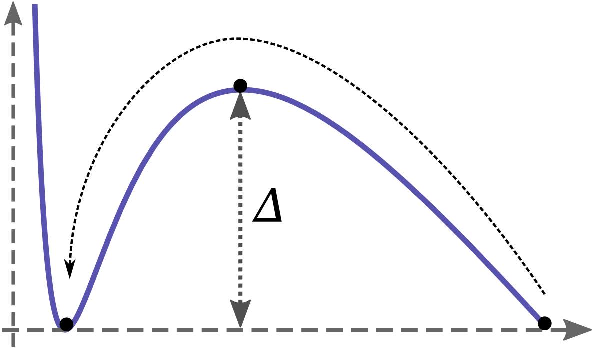
$$\mathscr{F}_{\kappa_\sharp}(\varrho) - \mathscr{F}_{\kappa_\sharp}(\varrho_\infty) \le -\frac{\varepsilon^3 \varrho_\infty}{3\beta}\int \Big(\sum_{k \in K^\delta} w_k\Big)^3 + O(\varepsilon^4).$$

The resonance condition $k^a = k^b + k^c$ ensures that

$$\int \Big(\sum_{k \in K^{\delta *}} w_k\Big)^3 > 0\ .$$

A mountain pass theorem

[Gvalani-S. '20]



Noise-induced transitions in $\mathbb{R}^d$

Start form deterministic gradient flow in $\mathbb{R}^d$

$$\dot{x}(t) = -\nabla F(x) \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^d$$

- F has two global minima $m_1, m_2 \in \mathbb{R}^d$.

Describe the particle transition from m_1 to m_2 under the influence of noise.

Modelproblem: Add Brownian motion

$$dX_t = -\nabla F(X_t)\,\mathrm{d}t + \sqrt{2\sigma} dB_t\,,$$

Question: Given $X(0) = m_1$, what is the probability that in some finite time $T > 0$, we have that $X(T) = m_2$ in the regime $\sigma \ll 1$?

Theorem (Freidlin–Wentzell)

The family of processes $\{X_t^\sigma\} \in C([0,T];\mathbb{R}^2)$ satisfy a LDP with good rate function $I : C([0,T];\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$

$$I(\gamma) = \frac{1}{4}\int_0^T |\dot{\gamma}(t) + \nabla F(\gamma(t))|^2 \,\mathrm{d}t.$$

and it holds

$$\mathrm{P}(X_t^\sigma \in \Gamma) \approx \exp\Big(-\sigma^{-1} \inf_{\gamma\in\Gamma} I(\gamma)\Big) \qquad \sigma \ll 1,$$

for any $\Gamma \subset C([0,T];\mathbb{R}^d)$.

Noise-induced transitions in $\mathbb{R}^d$

For $\gamma \in \Gamma = \{f \in C^1([0,T];\mathbb{R}^d) : \gamma(0) = m_1, \gamma(T) = m_2\}$ let $T^* = \arg\max_{t\in[0,T]}(F(\gamma(t)) - F(\gamma(0)))$:

$$\begin{aligned} I(\gamma) &\geq \frac{1}{4}\int_0^{T^*} |\dot\gamma(t) + \nabla F(\gamma(t))|^2 \,\mathrm{d}t = \frac{1}{4}\int_0^{T^*} |\dot\gamma(t) - \nabla F(\gamma(t))|^2 \,\mathrm{d}t + \int_0^{T^*} \dot\gamma(t)\cdot\nabla F(\gamma(t))\,\mathrm{d}t \\ &\geq F(\gamma(T^*)) - F(\gamma(0)) \geq \inf_{\gamma\in\Gamma}(F(\gamma(T^*)) - F(\gamma(0))) =: c - F(\gamma(0))\,, \end{aligned}$$

By classical mountain pass theorem: c a critical value of F, i.e., $\exists s \in \mathbb{R}^d : \nabla F(s) = 0, F(s) = c$.

$$\Rightarrow \qquad \mathrm{P}(X_t^\sigma \in \Gamma) \lesssim \exp(-\sigma^{-1}\Delta F) \qquad \text{where} \quad \Delta F = F(s) - F(m_1).$$

LDP for McKean-Vlasov interaction particle system

- Apply argument to the McKean–Vlasov N-particle system for $N \gg 1$

$$\mathrm{d}X_t^i = -\frac{\kappa}{N} \sum_{j=1, j\neq i}^{N} \nabla W(X^i - X^j)dt + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \dots, N$$

- [Dawson-Gärtner 1987] proved LDP with rate function for $\mu \in AC^2([0,T], \mathcal{P}_2(\mathbb{T}_L^d))$ given by

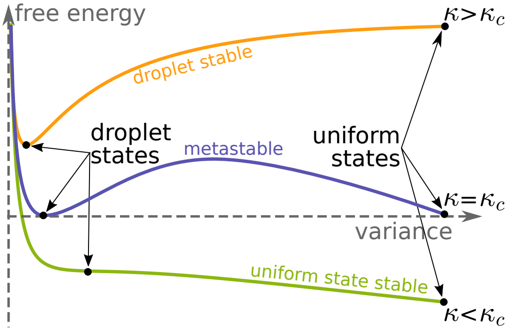
$$I_\kappa(\mu(\cdot)) := \frac{1}{4} \int_0^T \|\partial_t \mu_t - \nabla \cdot (\mu_t \nabla(\log \mu_t + \kappa W \star \mu_t))\|_{-1,\mu_t}^2 \,\mathrm{d}t$$

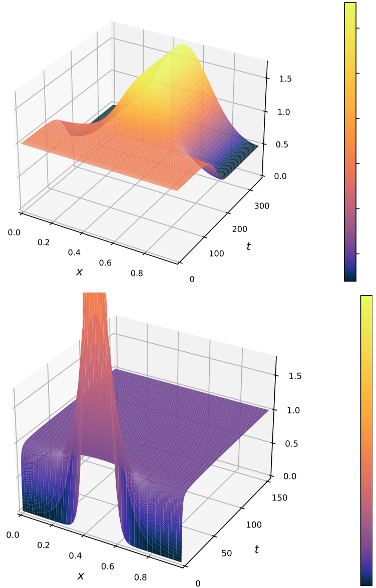
- Associated quasipotential to LDP is $\mathcal{F}_\kappa$!

$$\mathbb{P}(\text{transition: } \varrho_\infty \to \varrho_{\kappa_c}) \simeq \exp\Big(-N \inf\{I_\kappa(\mu(\cdot)) : \mu(0) = \varrho_\infty, \mu(T) = \varrho_{\kappa_c}\}\Big)$$
$$\leq \exp\left(-N \inf_\mu \left\{ \sup_{T^* \in [0,T]} (\mathcal{F}_\kappa(\mu(T^*)) - \mathcal{F}_\kappa(\mu(0))) : \mu(0) = \varrho_\infty, \mu(T) = \varrho_{\kappa_c} \right\}\right).$$

Discontinuous phase transitions and metastability

- N-particle system is metastable at disc. phase transition
- By [Dawson-Gärtner 1989] need to understand free energy





- Missing ingredient: mountain pass theorem for $\mathcal{F}_\kappa$

  Difficulties:
  - $(\mathcal{P}(\mathbb{T}_L^d), W_2)$ only metric space
  - $\mathcal{F}_\kappa$ only lower semicontinuous

A mountain pass theorem

Theorem [Gvalani-S. '20]

If $\mathcal{F}_{\kappa_c}$ has two distinct minimizers $\varrho_\infty \equiv 1/L^d$ and $\varrho_{\kappa_c} \in \mathcal{P}(\mathbb{T}^d_L)$, then there exists $\varrho^* \in \mathcal{P}(\mathbb{T}^d_L)$ distinct from ϱ_∞ and ϱ_{κ_c} such that $|\partial \mathcal{F}_{\kappa_c}|(\varrho^*) = 0$.

Moreover: $\mathcal{F}_{\kappa_c}(\varrho^*) = c$ with $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,Ts]} \mathcal{F}(\gamma(t))$,

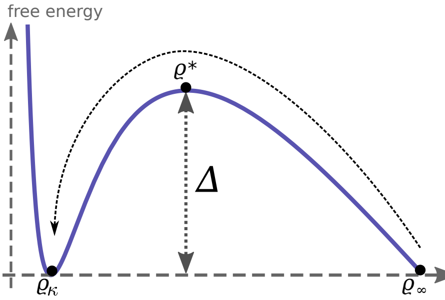
where $\Gamma = \{C([0,T];\mathcal{P}(\mathbb{T}^d_L)) : \gamma(0) = \varrho_\infty, \gamma(T) = \varrho_{\kappa_c}\}$.

Corollary (Arrhenius law)

The empirical McKean-Vlasov process $\varrho^{(N)}$ satisfies

$$\mathbb{P}\left[\varrho^N(T) \in \overline{B}^{W_2}_\varepsilon(\varrho_{\kappa_c}), \varrho^{(N)}(0) = \varrho_0^{(N)}\right] \lesssim e^{-N\Delta}$$

for N sufficiently large with $\mathbb{E}(W_2(\varrho_0^{(N)}, \varrho_\infty)) \to 0$ and $\Delta := \mathcal{F}_{\kappa_c}(\varrho^*) - \mathcal{F}_{\kappa_c}(\varrho_\infty)$ with ϱ^* the mountain pass point.



Models of nucleation and condensation

Nucleation of oversaturated vapor:

- Only monomers move
- no collisions between clusters
- clusters grow/shrink by one monomer

Applications:

- polymerization
- cloud and galaxy formation mechanism

[Smoluchowski 1916, Becker–Döring 1935]

Clustering of granular gases:

- Single beads hop to neighboring cells
- hopping rate depends on cell filling
- phases: frozen, clustering, gaseous

Applications:

- migration, population dynamics
- wealth exchange

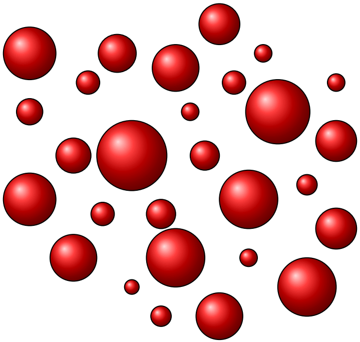
[Spitzer 1970 zero-range process]

Goal: Dynamic of cluster population.

[Tanaka et al J. Chem. Phys. 2011]

Longtime behavior of the exchange-driven growth model

[S. '19]



Exchange driven growth (EDG)

Previous models evolve size distribution $(X_k)_{k\geq 0}$ in a population of clusters [Ben-Naim, Krapivsky '03]

$$X_{k-1} + X_l \underset{K(k,l-1)}{\overset{K(l,k-1)}{\rightleftharpoons}} X_k + X_{l-1} \; , \qquad \text{for} \quad k,l \geq 1 \; .$$

- $K(l,k-1)$ rate kernel: jump of monomer from l to $k-1$ cluster
- rate equation is a countable nonlinear coupled birth-death chain:

$$\dot{c}_k \;=\; A_{k-1}[c]c_{k-1} - (A_k[c] + B_k[c])c_k + B_{k+1}[c]c_{k+1} \tag{EDG}$$

with (state-dependent) birth- and death-rates

$$A_{k-1}[c] = \sum_{l\geq 1} K(l,k-1)c_l \qquad \text{and} \qquad B_k[c] = \sum_{l\geq 1} K(k,l-1)c_{l-1} \qquad \text{for } k \geq 1.$$

- two conservation laws:

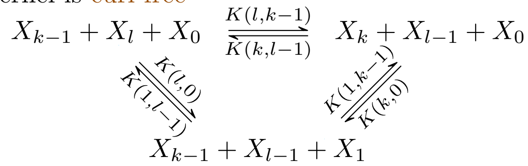
$$1 = \sum_{k\geq 0} c_k \qquad \text{and} \qquad \varrho = \sum_{k\geq 1} k c_k$$

Theorem (Well posedness [S. J Nonlinear Sci '19])

If the kernel K has at most linear growth $K(k,l) \leq C\,k\,l$, then the solution to (EDG) is a semigroup on $\mathcal{P}^\varrho = \{c \in \ell^1(\mathbb{N}_0) \;:\; c_k \geq 0 \;,\; \sum_{k\geq 0} c_k = 1 \;,\; \sum_{k\geq 1} k\,c_k = \varrho\}$ with $c(0) \in \mathcal{P}^\varrho$ for $\varrho \geq 0$.

Equilibria - longtime behavior - phase separation

If the kernel is curl-free



$$\Longleftrightarrow \quad K(k,l-1)\,K(l,0)\,K(1,k-1) = K(l,k-1)\,K(k,0)\,K(1,l-1)$$

then there exists a chemical potential and (formal) equilibria

$$Q_0 = 1, \quad Q_l = \prod_{k=1}^{l} \frac{K(1,k-1)}{K(k,0)} \quad \text{and} \quad \omega_l(\phi) = \frac{\phi^l Q_l}{Z(\phi)} \quad \text{with} \quad Z(\phi) = \sum_{l\geq 0} \phi^l Q_l$$

These equilibria have a critical mass density $\varrho_c \in [0,\infty]$

$$\varrho_c = \limsup_{\phi\uparrow\phi_c} \sum_{l\geq 1} l\omega_l(\phi) \qquad \text{with} \qquad \phi_c = \lim_{k\to\infty} \frac{K(k,0)}{K(1,k-1)} \in (0,\infty].$$

$\Rightarrow$ Free energy functional (gradient flow, Lyapunov function)

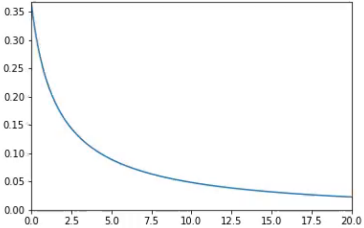
$$\mathcal{F}[c] = \sum_{k\geq 0} c_k \log \frac{c_k}{Q_k} = \sum_{k\geq 0} c_k \log c_k + \sum_{k\geq 0} c_k \log \frac{1}{Q_k}.$$

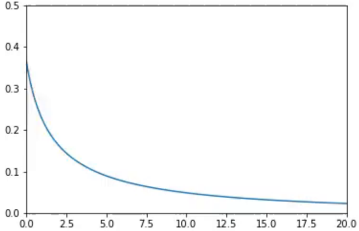
Longtime behavior and phase separation

Theorem [S. J Nonlinear Sci '19]

Let K be curl-free, sublinear, sufficiently regular with $\rho_c \in (0,\infty)$.
Then for any $\varrho \in (0,\infty)$ and any $c(0) \in \mathcal{P}^\varrho$, the solution c of (EDG) satisfies

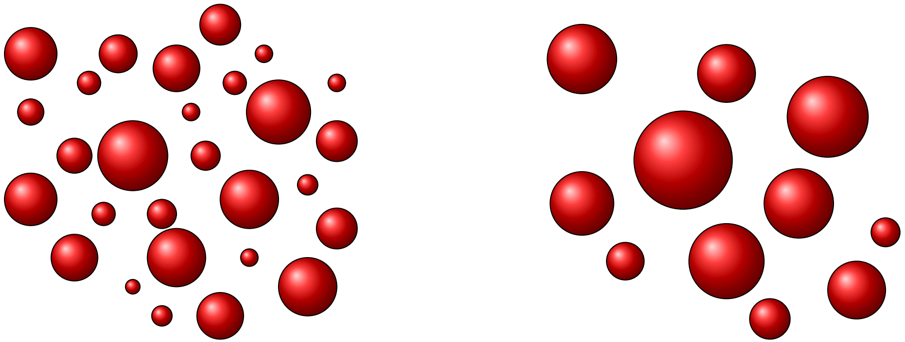
1. If $\varrho \le \varrho_c$: Then $\mathcal{F}[c(t)] \to \mathcal{F}[\omega^\varrho]$ and $\sum_{l\ge 0}(l+1)|c_l(t) - \omega_l^\varrho| \to 0$ as $t \to \infty$.
2. If $\varrho > \varrho_c$: Then $\mathcal{F}[c(t)] \to \mathcal{F}[\omega^{\varrho_c}] + (\varrho - \varrho_c)\log\phi_c$ and $c_l(t) \to \omega_l^{\varrho_c}$ for all $l \ge 0$ as $t \to \infty$.
   In particular no ℓ^1-convergence, since excess mass $\varrho - \varrho_c$ vanishes!

1 

2 

Self-similar behavior of the exchange-driven growth model

[Eichenberg-S. '21]



Special class of kernels [Ben-Naim, Krapivsky '03]

Product kernel: for $\lambda \in [0, 2)$

$$K(k,l) = K_\lambda(k,l) = a_\lambda(k)a_\lambda(l)$$

$$\text{with} \quad a_\lambda(k) = \begin{cases} k^\lambda, & \lambda > 0, \\ 1 - \delta_{k,0}, & \lambda = 0, \end{cases}$$

$$\begin{cases} \dot c_0 = M_\lambda[c]\, c_1, \\ \dot c_1 = M_\lambda[c]\, \big(-2c_1 + 2^\lambda c_2\big), \\ \dot c_k = M_\lambda[c]\, \big((k-1)^\lambda c_{k-1} - 2k^\lambda c_k + (k+1)^\lambda c_{k+1}\big). \end{cases} \qquad \text{EDG}_\lambda$$

Theorem (Coarsening rate)

$$\ell(t) \propto \begin{cases} t^\beta, & \text{if } 0 \le \lambda < 3/2, \\ \exp(C_0 t), & \text{if } \lambda = 3/2, \\ (t_{\text{gel}} - t)^\beta, & \text{if } 3/2 < \lambda < 2, \end{cases}$$

- $K_\lambda(k,0) = 0$ for all $k \ge 1$
  $\Rightarrow$ no formation of new clusters
- Homogeneity and symmetry simplify (EDG) through moments

$$M_\kappa[c] = \sum_{l \ge 1} l^\kappa\, c_l$$

- Average size of alive clusters

$$\ell(t) = \frac{1}{1 - c_0(t)} \sum_{k=1}^{\infty} k\, c_k(t) = \frac{\rho}{M_0[c]},$$

- Coarsening exponent

$$\beta = \frac{1}{3 - 2\lambda}.$$

Self-similar behavior

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Behave solutions dynamical self-similar?

$$c_k(t) \propto \rho\, s(t)^{-2} g_\lambda\big(s(t)^{-1} k\big) \qquad \text{for } t \gg 1?$$

for the explicit profile

$$g_\lambda(x) = \frac{1}{Z_\lambda} \frac{x^{1-\lambda}}{2-\lambda} \exp\left(-\frac{x^{2-\lambda}}{(2-\lambda)^2}\right),$$

Theorem (Self-similar behavior)

1. For $0 \le \lambda < 3/2$ there exists $C = C(\lambda, \rho) > 0$ and
$$s(t) = Ct^\beta \qquad \text{with } \beta = (3-2\lambda)^{-1},$$
such that every global solution c to (EDG$_\lambda$) satisfies
$$\mu_c(t) \rightharpoonup \rho\, g_\lambda \qquad \text{as } t \to \infty.$$
2. For $\lambda = 3/2$ ...
3. For $3/2 < \lambda < 2$ and t_{gel} as before there exists $C = C(\lambda, \rho) > 0$ and
$$s(t) = C(t_{\text{gel}} - t)^\beta \qquad \text{with } \beta = (3-2\lambda)^{-1},$$
such that every solution c to (EDG$_\lambda$) existing on $[0, t_{\text{gel}})$ satisfies $\mu_c(t) \rightharpoonup \rho\, g_\lambda$ as $t \to t_{\text{gel}}$.

Measure-valued formulation

$$\mu_c(t) = s(t) \sum_{k \ge 1} c_k(t) \delta_{s(t)^{-1} k}.$$

Number of clusters

$$M_0[c] = 1 - c_0 = s^{-1}(t) \int_0^\infty \mathrm{d}\mu_c.$$

Total mass density

$$M_1[c] = \int_0^\infty x\, \mathrm{d}\mu_c.$$

Some ingredients for the proof

Time-change

$$\tau(t) = \int_0^t M_\lambda[c](s)\,\mathrm{d}s.$$

Discrete weighted Laplacian:

$$\begin{cases} \partial_\tau u = \Delta_{\mathbb{N}}(a_\lambda u), & k \geq 1, \\ u(\tau,0) = 0, & \tau \geq 0. \end{cases}$$

Tail-distribution: $U(t,k) = \sum_{l \geq k} u(t,l)$

$$\begin{cases} \partial_t U(k) = \partial^-(a_\lambda \partial^+ U)(k), \\ \partial^+ U(t,0) = 0. \end{cases}$$

Proposition (Discrete Nash-inequality)

$$\|U\|_2^2 \lesssim \|U\|_1^{\frac{2(2-\lambda)}{3-\lambda}} E_\lambda(U)^{\frac{1}{3-\lambda}},$$

Dirichlet form: $E_\lambda(U) = \sum_{k\geq 1} k^\lambda |\partial^+ U(k)|^2$.

$\Rightarrow$ Nash-continuity, decay, moment bounds, ...

Continuous solution:

$$\begin{cases} \partial_t \mathcal{U} = \partial_x(a_\lambda \partial_x \mathcal{U}) = \mathcal{L}_\lambda \mathcal{U}, & (t,x) \in \mathbb{R}_+^2, \\ a_\lambda \partial_x \mathcal{U}|_{x=0} = 0, & t \in \mathbb{R}_+, \\ \mathcal{U}(0,\cdot) = \mathcal{U}_0, & x \in \mathbb{R}_+. \end{cases}$$

Discrete-to-continuum interpolation:

$$\mathcal{U}_\varepsilon(t,x) = \varepsilon^{-\alpha} U(\varepsilon^{-1} t, \lfloor \varepsilon^{-\alpha} x \rfloor + 1).$$

parabolic scaling: $k \propto t^\alpha$ with $\alpha = \frac{1}{2-\lambda} \in \left[\frac{1}{2}, \infty\right)$

Theorem

It holds $\mathcal{U}_\varepsilon \to \mathcal{U}$.

Proof: Replacement lemma, compactness

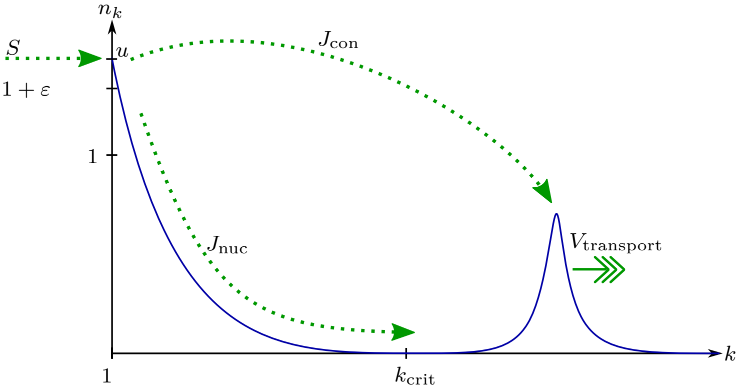
Note: For $t = 1$, we get

$$\varepsilon^{-\alpha} U(\varepsilon^{-1}, \lfloor \varepsilon^{-\alpha} x \rfloor + 1) \to \rho\, \mathcal{G}_\lambda(x).$$

$\Rightarrow$ Obtain long-time behavior by setting $t = \varepsilon^{-1}$.

Oscillations in a Becker–Döring model with injection and depletion

[Niethammer-Pego-S.-Velazquez, to appear SIAP, arXiv:2102.06751]



Hopf bifurcation of a bubbleator

Becker-Döring bubbleator:

$$\emptyset \xrightarrow{S} X_1 \qquad \text{Injection } S > 0$$

$$X_1 + X_k \underset{k^\alpha(1+k^{-\gamma})}{\overset{k^\alpha}{\rightleftharpoons}} X_{k+1}\,, \qquad k = 1, 2, 3, \ldots$$

$$X_k \xrightarrow{Rk^r} \emptyset, \qquad \text{Depletion } R > 0.$$

Limit model (small oversaturation)

$$\partial_\tau u = 1 - \int_0^\infty f(x,\tau) x^\alpha \, dx\,,$$

$$\partial_\tau f + \partial_x(x^\alpha f) = -\bar{R} x^r f\,, \qquad x > 0$$

$$x^\alpha f(x,\tau) \to e^{u(\tau)} \qquad x \to 0$$

Parameters: $\bar{R} = 0.1$, $\alpha = \beta = 1/3$, $r = 2/3$