Quantitative particle approximation of nonlinear Fokker-Planck equations with singular kernels

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joint work with C. Olivera (Campinas) and A. Richard (CentraleSupélec) to appear in Ann Sc Norm Super Pisa’21+

Workshop METASTABILITY, MEAN-FIELD PARTICLE SYSTEMS AND NON LINEAR PROCESSES
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Overview

Introduction

Rate of convergence to the PDE

The nonlinear process and propagation of chaos

Recent progress and perspectives
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Rate of convergence to the PDE

The nonlinear process and propagation of chaos

Recent progress and perspectives
We study on \([0, T) \times \mathbb{R}^d\)

\[
\begin{aligned}
\partial_t u(t, x) &= \Delta u(t, x) - \nabla \cdot \left( u(t, x) K * x u(t, x) \right), \\
u(0, x) &= u_0(x),
\end{aligned}
\]  

(NLFP)

with \(K\) \textbf{locally integrable} and \textbf{singular} at 0.

Our main interest: stochastic particle approximation of (NLFP).

Why?

- Macroscopic to microscopic description (and back!);
- Numerical schemes...
Classical approach: mean-field interactions

- (NLFP) is seen as the FP equation for the non-linear process

\[
\begin{cases}
    dX_t = K * u_t(X_t) \, dt + \sqrt{2} \, dW_t, \\
    \mathcal{L}(X_t) = u_t.
\end{cases}
\]  

(NLSDE)

- Particle system in mean-field interaction:

\[
dX_{t,N}^i = \frac{1}{N} \sum_{j=1}^{N} K(X_{t,N}^i - X_{t,N}^j) + \sqrt{2} \, dW_{t,N}^i
\]  

and its empirical measure \( \mu_{N} \cdot = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t,N}^i} \).

- Propagation of chaos: convergence in law of \( \mu_{N} \) in \( \mathcal{P}(C) \) towards the law of \( X \).

- Issues: \( K \) singular \( \rightarrow \) wellposedness of (PS), (NLSDE) and the propagation of chaos?

- Probabilistic approach to singular interactions:
  - Boltzmann, Burgers, Navier-Stokes, Keller-Segel equations, ...
  - studied by Bossy, Calderoni, Cattiaux, Fournier, Hauray, Jabin, Jourdain, Méléard, Mischler, Osada, Pulvirenti, Talay, ...
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(NLSDE)

- Particle system in mean-field interaction:

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dX_{t,N}^{i,N} = \frac{1}{N} \sum_{j=1}^{N} K(X_{t,N}^{i,N} - X_{t,N}^{j,N}) + \sqrt{2}dW_{t,N}^{i,N}
\]

(PS)

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Another viewpoint: moderate interaction

Motivated by singular attractive kernels for which either:

- existence of (PS) is unknown;
- or existence ok, but convergence unknown.

Moderately interacting particles:

\[ dX_{t}^{i,N} = F \left( V_{N} \ast (K \ast \mu_{t}^{N})(X_{t}^{i,N}) \right) dt + \sqrt{2}dW_{t}^{i,N}, \]

where:

- \( V_{N}(x) = N^{d \alpha} V(N^{\alpha}x), \alpha > 0 \) and \( V \) a regular density;
- \( F \) a smooth cut-off.

Some references: [Oelschläger'85], [Méléard & Roelly'87], [Jourdain & Méléard’98]

→ A semigroup approach was recently developed by Flandoli, Olivera and their collaborators to get uniform (non-quantitative) convergence of \( V_{N} \ast \mu_{t}^{N} \) towards a mild solution to:

\textit{FKPP, 2d Navier-Stokes equations, PDE-ODE system related to aggregation phenomena.}
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Our main objectives

What assumptions on $K$ and what suitable functional framework for (NLFP) so the following holds?

- Convergence of \( \{\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i,N}, t \in [0, T]\} \) to the solution (NLFP) when $N \to \infty$:
  - which range of $\alpha$?
  - what is the rate of convergence?

- Well-posedness of (NLSDE).

- Propagation of chaos towards (NLSDE) (without the cut-off and the mollifier)
Which kernels?

A typical example in dimension $d \geq 2$ is the family of **Riesz kernels**:

$$K_s(x) = \pm \nabla \mathcal{V}_s(x)$$

where

$$\mathcal{V}_s(x) := \begin{cases} |x|^{-s} & \text{if } s \in (0, d-1) \\ -\log |x| & \text{if } s = 0 \end{cases}, \quad x \in \mathbb{R}^d.$$

Examples:

- **Coulomb interactions**: $K_s$, with $s = d - 2$ ($d \geq 3$);
- **2d Navier-Stokes equation (vorticity)**: $K(x) = \frac{x \perp}{|x|^2}$;
- **Parabolic-elliptic Keller-Segel model**: $K(x) = -\chi \frac{x}{|x|^d}$ (**attractive**...);
- **Some attractive-repulsive kernels.**
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Assumptions on $K$ and $\alpha$

($H_K$):

(i) $K \in L^p(B_1)$, for some $p \in [1, +\infty]$;
(ii) $K \in L^q(B_1^c)$, for some $q \in [1, +\infty]$;
(iii) There exists $r \geq \max(p', q')$ and $\zeta \in (0, 1]$ such that:

$$N_\zeta(K * f) \lesssim \|f\|_{L^1 \cap L^r(\mathbb{R}^d)}, \quad \forall f \in L^1 \cap L^r(\mathbb{R}^d).$$

($N_\zeta$ the Hölder seminorm of parameter $\zeta$.)

($H_\alpha$): $\alpha$ and $r$ satisfy

$$0 < \alpha < \frac{1}{d + 2d(\frac{1}{2} - \frac{1}{r}) \vee 0}.$$
Assumptions on $K$ and $\alpha$

$(H_K)$:

(i) $K \in L^p(B_1)$, for some $p \in [1, +\infty]$;
(ii) $K \in L^q(B_1^c)$, for some $q \in [1, +\infty]$;
(iii) There exists $r \geq \max(p', q')$ and $\zeta \in (0, 1]$ such that:

$$\mathcal{N}_\zeta(K \ast f) \lesssim \|f\|_{L^1 \cap L^r(\mathbb{R}^d)}, \quad \forall f \in L^1 \cap L^r(\mathbb{R}^d).$$

($\mathcal{N}_\zeta$ the Hölder seminorm of parameter $\zeta$.)

$(H_\alpha)$: $\alpha$ and $r$ satisfy

$$0 < \alpha < \frac{1}{d + 2d\left(\frac{1}{2} - \frac{1}{r}\right) \vee 0}.$$  

Notice here that for $f \in L^1 \cap L^r(\mathbb{R}^d)$ one has

$$\|K \ast f\|_{L^\infty(\mathbb{R}^d)} \leq C_{K,d} \|f\|_{L^1 \cap L^r(\mathbb{R}^d)}.$$
Assume (always today) that \( u_0 \geq 0 \) and \( \int_{\mathbb{R}^d} u_0 = 1 \).

**Proposition**

Let \( K \) satisfying \((H_K)\)(i)-(ii), \( u_0 \in L^1 \cap L^r(\mathbb{R}^d) \) with \( r \geq \max(p', q') \).

There exists \( T > 0 \) and a unique \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) s.t.

\[
   u \in C([0, T]; L^1 \cap L^r(\mathbb{R}^d))
\]

and

\[
   u_t = e^{t\Delta} u_0 - \int_0^t \nabla \cdot (e^{(t-s)\Delta}(u_s K * u_s)) \, ds, \quad 0 \leq t \leq T.
\]

Denote by \( T_{\text{max}} \) the maximal existence time.
Convergence of the mollified empirical measure

**Theorem ([O.-R.-T. Ann Sc Norm Super Pisa’21+])**

Let $T < T_{\text{max}}$ and assume $(H_K)$ and $(H_\alpha)$. Under suitable conditions on the initial conditions, we have for $\{u_t^N = V_t^N * \mu_t^N, t \in [0, T]\}_{N \in \mathbb{N}}$: for all $\epsilon > 0$ and all $m \geq 1$,

$$\|u^N - u\|_{T,L^1 \cap L^r(\mathbb{R}^d)} \lesssim \|u_0^N - u_0\|_{L^1 \cap L^r(\mathbb{R}^d)} \lesssim \|u_0^N - u_0\|_{L^1 \cap L^r(\mathbb{R}^d)} \lesssim \|u_0^N - u_0\|_{L^1 \cap L^r(\mathbb{R}^d)} \lesssim$$

$$\|u_0^N - u_0\|_{L^1 \cap L^r(\mathbb{R}^d)} \lesssim + N^{-\rho + \epsilon},$$

where

$$\rho = \min \left( \alpha \zeta, \frac{1}{2} \left( 1 - \alpha (d + \alpha (d(1 - \frac{2}{r}) \lor 0) \right) \right).$$
Convergence of the mollified empirical measure

Without the cutoff $F$ in the drift of the particles, we get:

**Corollary**

For any $\varepsilon \in (0, \varrho)$, any $\eta > 0$ and any $m \geq 1$,

$$
\mathbb{P} \left( \| u_t^N - u_t \|_{T, L^1 \cap L^r(\mathbb{R}^d)} \geq \eta \right) \lesssim \frac{1}{\eta^m} \left( \| u_0^N - u_0 \|_{L^1 \cap L^r(\mathbb{R}^d)} \right)_{L^m(\Omega)} + N^{-\varepsilon + \varepsilon}^m.
$$
Some consequences and remarks

- Same rate for the **genuine empirical measure** of (PS)

\[
\left\| \sup_{t \in [0, T]} \left\| \mu_t^N - u_t \right\| \right\|_{L^m(\Omega)} \leq C N^{-\theta + \varepsilon},
\]

where \( \| \cdot \|_0 \) denotes the Kantorovich-Rubinstein metric.

- The rate in the previous results holds almost surely.

- Cannot expect here a \( N^{-\frac{1}{2}} \) rate of convergence because of the short range interactions: “best possible” \( N^{-\alpha} \).


Applications

- **Coulomb-type kernels** (like Biot-Savart in \( d = 2 \), Riesz with \( s = d - 2 \)),
  - convergence happens for \( \alpha < \frac{1}{2(d-1)} \) (\( d = 2 \) \( \Rightarrow \) \( \alpha = \left( \frac{1}{2} \right)^{-} \));
  - best possible rate is \( \varrho = \left( \frac{1}{2(d+1)} \right)^{-} \) (obtained for \( \alpha = \left( \frac{1}{2(d+1)} \right)^{+} \), \( r = +\infty \), \( \zeta = 1 \)).

- **Keller-Segel parabolic elliptic** (\( d = 2 \) : global solution \( \chi < 8\pi \), blow up in finite time otherwise).
  - we get the above rate for any value of \( \chi \);
  - the result holds even if the PDE explodes in finite time (\( \chi > 8\pi \)).

- **The Riesz kernels with** \( s > d - 2 \) do not satisfy \((H_K)(iii)\).
  However, by imposing more regularity on the initial conditions and smaller values of \( \alpha \), we get a rate of convergence for singular Riesz kernels with \( s \in (d - 2, d - 1) \).
Applications

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Sketch of proof

Derive the SPDE satisfied by the mollified empirical measure $u_N$ in its mild form

$$u_t^N(x) = e^{t \Delta} u_0^N(x) - \int_0^t \nabla \cdot e^{(t-s) \Delta} \langle \mu_s^N, V^N(x - \cdot) F(K * u_s^N(\cdot)) \rangle \, ds$$

$$- \frac{1}{N} \sum_{i=1}^N \int_0^t e^{(t-s) \Delta} \nabla V^N(x - X_{s,i}^N) \cdot dW_s^i, \quad x \in \mathbb{R}^d$$

applying Itô’s formula to $G_{t,V^N}(s, x - \cdot) := e^{(t-s) \Delta} V^N(x - \cdot)$ on each particle between 0 and $t$ for $x$ and $t$ fixed. (sum up, $\frac{1}{N}$, rearrange..)

For $q \geq 1$ establish that

$$\sup_{N \in \mathbb{N}^*} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| u_t^N \right\|_{L^q(\mathbb{R}^d)}^q \right] < \infty.$$ 

(using the above mild form and combining Gronwall lemma and "martingale" estimate)
Sketch of proof

▶ Derive the SPDE satisfied by the mollified empirical measure $u^N$ in its mild form

$$u^N_t(x) = e^{t\Delta}u^N_0(x) - \int_0^t \nabla \cdot e^{(t-s)\Delta} \langle \mu^N_s, V^N(x - \cdot) F(K * u^N_s(\cdot)) \rangle \, ds$$

$$- \frac{1}{N} \sum_{i=1}^N \int_0^t e^{(t-s)\Delta} \nabla V^N(x - X^i_s, N) \cdot dW^i_s, \quad x \in \mathbb{R}^d$$

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▶ For $q \geq 1$ establish that

$$\sup_{N \in \mathbb{N}^*} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| u^N_t \right\|^q_{L^r(\mathbb{R}^d)} \right] < \infty.$$

(Using the above mild form and combining Gronwall lemma and "martingale" estimate)
Decompose $\|u_t^N - u_t\|_{L^1 \cap L^r}$ into several terms ($u$ is the unique sln of the mild equation with cut-off):

$$u_t^N(x) - u_t(x) = e^{t\Delta} (u_0^N - u_0)(x) + E_t(x) - M_t^N(x)$$

$$+ \int_0^t \nabla \cdot e^{(t-s)\Delta} \left( u_s F(K \ast u_s) - u_s^N F(K \ast u_s^N) \right)(x) \, ds,$$

where we have set

$$E_t(x) := \int_0^t \nabla \cdot e^{(t-s)\Delta} \left( \mu_s^N, V_s^N (x - \cdot) \left( F(K \ast u_s^N (x)) - F(K \ast u_s^N (\cdot)) \right) \right) \, ds,$$

$$M_t^N(x) := \frac{1}{N} \sum_{i=1}^N \int_0^t e^{(t-s)\Delta} \nabla V_s^N (x - X_{s}^{i,N}) \cdot dW_s^i. \quad (1)$$
Pivoting in the last term and using that

\[ \| K \ast f \|_{L^\infty(\mathbb{R}^d)} \leq C_{K,d} \| f \|_{L^1 \cap L^r(\mathbb{R}^d)} \]

\[ \| u_t^N - u_t \|_{L^1 \cap L^r(\mathbb{R}^d)} \leq \| u_0^N - u_0 \|_{L^1 \cap L^r(\mathbb{R}^d)} + C \int_0^t \frac{1}{\sqrt{t-s}} \| u_s^N - u_s \|_{L^1 \cap L^r(\mathbb{R}^d)} ds \]

\[ + \| E_t \|_{L^1 \cap L^r(\mathbb{R}^d)} + \| M_t^N \|_{L^1 \cap L^r(\mathbb{R}^d)}. \]

Using the Grönwall lemma for convolution integrals, we obtain

\[ \| u_t^N - u_t \|_{t,L^1 \cap L^r(\mathbb{R}^d)} \leq C \left( \| u_0^N - u_0 \|_{L^1 \cap L^r(\mathbb{R}^d)} \right. \]

\[ + \| E \|_{t,L^1 \cap L^r(\mathbb{R}^d)} + \| M_t^N \|_{t,L^1 \cap L^r(\mathbb{R}^d)} \right). \]

(2)

It remains to control the moments of \( \| E \|_{t,L^q(\mathbb{R}^d)} \) and \( \| M \|_{t,L^q(\mathbb{R}^d)} \) for \( q \in \{1, r\} \). This is where the two expression in the definition of the speed \( \rho \) appear.
Observe that (positivity of $F$, heat kernel)

$$\|E_t\|_{L^q(\mathbb{R}^d)} \leq C \int_0^t \frac{1}{(t-s)^\frac{1}{2}} \left( \int_{\mathbb{R}^d} \langle \mu_s^N, V^N(x-\cdot) \rangle^{\frac{1}{q}} dx \right)^\frac{1}{q} ds.$$

$F$ Lipschitz and the $\zeta$-Hölder continuity of $K \ast u^N$ give

$$\|E_t\|_{L^q(\mathbb{R}^d)} \leq C \int_0^t \|u_s^N\|_{L^1 \cap L^r(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} \langle \mu_s^N, V^N(x-\cdot) | \cdot - x \rangle^{\zeta} \right)^\frac{1}{q} ds.$$

Since $V$ is compactly supported (wlog, assume $\text{supp}(V) \subset B_1$), we have that $V^N(x-y) |y-x|^{\zeta} \leq N^{-\alpha \zeta} V^N(x-y)$. Thus

$$\|E_t\|_{L^q(\mathbb{R}^d)} \leq \frac{C}{N^{\alpha \zeta}} \int_0^t \frac{1}{(t-s)^\frac{1}{2}} \|u_s^N\|_{L^1 \cap L^r(\mathbb{R}^d)} \|u_s^N\|_{L^q(\mathbb{R}^d)} ds$$

$$\leq \frac{C}{N^{\alpha \zeta}} \int_0^t \frac{1}{(t-s)^\frac{1}{2}} \|u_s^N\|^2_{L^1 \cap L^r(\mathbb{R}^d)} ds,$$
Apply Hölder’s inequality with $p = \frac{3}{2}$ to obtain
\[
\|E_t\|_{L^q(\mathbb{R}^d)} \leq \frac{C}{N\alpha\zeta} \left( \int_0^t (t - s)^{-\frac{3}{4}} \, ds \right)^{\frac{2}{3}} \left( \int_0^t \|u_s^N\|_{L^1 \cap L^r(\mathbb{R}^d)}^6 \, ds \right)^{\frac{1}{3}}.
\]

Finally, we have from Jensen’s inequality that for $m \geq 3$,
\[
\|\|E\|_{t,L^q(\mathbb{R}^d)}\|_{L^m(\Omega)} \leq \frac{C}{N\alpha\zeta} \left( \int_0^t \mathbb{E} \left[ \|u_s^N\|_{L^1 \cap L^r(\mathbb{R}^d)}^{2m} \right] \, ds \right)^{\frac{1}{m}}
\]
and the bound uniform bounds on $u^N$ permits to conclude that
\[
\|\|E\|_{t,L^q(\mathbb{R}^d)}\|_{L^m(\Omega)} \leq \frac{C}{N\alpha\zeta}.
\] (3)

This inequality immediately extends to $1 \leq m < 3$. 
Main issue: control the moments of

$$\sup_{t \leq T} \left\| \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} e^{(t-s)\Delta} \nabla V^N (X_s^i - \cdot) dW^i_s \right\|_{L^1 \cap L^r(\mathbb{R}^d)}.$$

- Not a martingale, fix the time in the heat operator and it becomes one;
- to control the $L^1 \cap L^r(\mathbb{R}^d)$ norm, use stochastic integration techniques in infinite-dimensional spaces [van Neerven et al.'07];
- use Garsia-Rodemich-Rumsey’s lemma to put the sup inside.

Note that this is where the limitation on $\alpha (H_\alpha)$ arises.
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Definition

Assume \( (H_K)(i)-(ii) \), \( u_0 \in L^1 \cap L^r(\mathbb{R}^d) \) with \( r \geq \max(p', q') \) and consider the canonical space \( C([0, T]; \mathbb{R}^d) \) equipped with its canonical filtration. We say that \( Q \in \mathcal{P}(C([0, T]; \mathbb{R}^d)) \) solves the nonlinear martingale problem associated to (NLSDE) if:

(i) \( Q_0 = u_0 \);

(ii) For any \( t \in (0, T] \), \( Q_t \) has density \( q_t \) and \( q \in C([0, T]; L^1 \cap L^r(\mathbb{R}^d)) \);

(iii) For any \( f \in C^2_c(\mathbb{R}^d) \), the process \( (M_t)_{t \in [0,T]} \) defined as

\[
M_t := f(w_t) - f(w_0) - \int_0^t \left[ \Delta f(w_s) + \nabla f(w_s) \cdot (K \ast q_s(w_s)) \right] ds
\]

is a \( Q \)-martingale, where \( (w_t)_{t \in [0,T]} \) denotes the canonical process.
The nonlinear process

**Proposition**

Let $T < T_{max}$. Then the martingale problem related to (NLSDE) is well-posed.

⇒ the McKean-Vlasov equation (NLSDE) admits a unique weak solution $\tilde{X}$. Combined with the convergence theorem, it comes:

$$\max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |X_{t, N}^i - \tilde{X}_t^i|_{L^m(\Omega)} \leq CN^{-\theta+\varepsilon}, \quad \forall N \in \mathbb{N}^*. $$
Let $T < T_{\text{max}}$. Then the martingale problem related to (NLSDE) is well-posed.

⇒ the McKean-Vlasov equation (NLSDE) admits a unique weak solution $\tilde{X}$. Combined with the convergence theorem, it comes:

$$\max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |X_{t, N}^i - \tilde{X}_t^i|_{L^m(\Omega)} \leq C N^{-\varrho + \varepsilon}, \quad \forall N \in \mathbb{N}^*.$$
Propagation of chaos

**Theorem**

Same hypotheses as in previous Theorem + Assume that \( \{X^i_0, \ i \in \mathbb{N}\} \) are identically distributed and that \( \langle u^N_0, \varphi \rangle \overset{\mathbb{P}}{\to} \langle u_0, \varphi \rangle, \ \forall \varphi \in C_b(\mathbb{R}^d) \).

Then

\[
\mu^N \overset{(d)}{\to} Q,
\]

where \( Q \) is the law of the solution of (NLSDE).

Example: 2d Keller-Segel parabolic-elliptic equation. We obtain local existence and uniqueness of (NLSDE) for all values of \( \chi \) and the propagation of chaos towards it.
Theorem

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Example: 2d Keller-Segel parabolic-elliptic equation. We obtain local existence and uniqueness of (NLSDE) for all values of \( \chi \) and the propagation of chaos towards it.
Sketch of proof

Usual strategy:
1. Consider the nonlinear MP with cutoff;
2. Prove the tightness of the family $\Pi^N := \mathcal{L}(\mu^N)$ in the space $\mathcal{P}(\mathcal{P}(\mathcal{C}([0, T]; \mathbb{R}^d)))$;
3. Prove that any limit point $\Pi^\infty$ of $\Pi^N$ is $\delta_Q$.
4. Lift the cutoff.

(3) is the most technical part:
- work in fact on $\mathcal{H} := \mathcal{P}(\mathcal{C}([0, T]; \mathbb{R}^d)) \times \mathcal{C}([0, T]; L^1 \cap L^r(\mathbb{R}^d))$ with $\tilde{\Pi}^N = \mathcal{L}(\mu^N, u^N)$ (as in [Méléard & Roelly'87]).
- introduce a quadratic functional $\Gamma$ on $\mathcal{H}$, which depends on the form of the martingale problem.
- use the convergence of $\tilde{\Pi}^N$ to $\tilde{\Pi}^\infty$ to prove that $\Gamma = 0$ $\tilde{\Pi}^\infty$-a.e.
  This is where $\mu^N$ and the particle system appear.
- deduce that the first coordinate of $\tilde{\Pi}^\infty$ solves the nonlinear MP (with cutoff).
Sketch of proof

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Overview

Introduction

Rate of convergence to the PDE

The nonlinear process and propagation of chaos

Recent progress and perspectives
Recent progress

▶ In a recent work [Guo & Luo '21] extend our method to particles with common noise:

\[
dX_{t}^{i,N} = V^{e} \ast (K \ast \mu_{t}^{N})(X_{t}^{i,N})dt + \sqrt{2\nu} \sum_{k} \sigma_{k}^{N}(X_{t}^{i,N})dW_{t}^{k},
\]

and quantify its convergence (in a two step procedure) to

\[
\partial_{t}u(t, x) = \nu \Delta u(t, x) - \nabla \cdot (u(t, x) K \ast x u(t, x))
\]

for a class of kernels such as repulsive Riesz kernels for \( s \in [0, d - 2] \).

▶ Rate of convergence for the IPS without cutoff, working on the torus:

\[
\limsup_{N \to +\infty} N^{\rho - \varepsilon} \sup_{t \in [0, T]} \|u_{t}^{N} - u_{t}\|_{L^{p}(\mathbb{T}^{d})} \leq X \ a.s.
\]

▶ Extension to Burgers.
Some next steps

- Numerical applications: use our result to quantify the convergence of a scheme coming from the moderately interacting particles.

- Treat non-Markovian particle systems: e.g. the parabolic-parabolic Keller-Segel model.

- Improve the constraint on $\alpha$ by changing the functional space.
P. Cattiaux and L. Pédèches.  
The 2-D stochastic Keller-Segel particle model: existence and uniqueness.  

E. Cépa and D. Lépingle.  
Diffusing particles with electrostatic repulsion.  

Uniform convergence of proliferating particles to the FKPP equation.  

F. Flandoli, C. Olivera, and M. Simon.  
Uniform approximation of 2 dimensional Navier-Stokes equation by stochastic interacting particle systems.  

N. Fournier and B. Jourdain.  
Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes.  

C. Guo, and D. Luo.  
Scaling Limit of Moderately Interacting Particle Systems with Singular Interaction and Environmental Noise.  
B. Jourdain and S. Méléard.
Propagation of chaos and fluctuations for a moderate model with smooth initial data.

C. Marchioro and M. Pulvirenti.
Hydrodynamics in two dimensions and vortex theory.

S. Méléard and S. Roelly-Coppoletta.
A propagation of chaos result for a system of particles with moderate interaction.

K. Oelschläger.
A law of large numbers for moderately interacting diffusion processes.

Particle approximation of the 2-d parabolic-elliptic Keller-Segel system in the subcritical regime.

Quantitative particle approximation of nonlinear Fokker-Planck equations with singular kernel.
J. M. A. M. van Neerven, M. C. Veraar, and L. Weis.
Stochastic integration in UMD Banach spaces.