Gradient Flow Structure of the Landau Equation

Metastability, Mean-Field Particle Systems and Non Linear Processes

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May 17, 2022

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Plasma

- Plasma is the fourth state of matter - formed by heating a gas at high temperatures (3,000 - 10,000 K) so electrons in a gas are separated from nuclei.
- Examples include stars, lightning, aurora borealis, interstellar matter, fluorescent lights, neon signs, etc...
- Main focus of the ITER project is to produce **net energy** through fusion - **four times** more energy than nuclear, **four million times** more than coal, oil, or gas.
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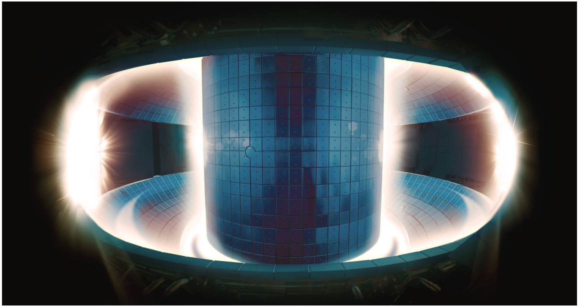

Figure: KSTAR tokamak plasma image from the National Fusion Research Institute of Korea

Landau Equation (written by Landau in 1936 [LP76])

For $\gamma \in [-3, 1)$, evolution of density, f, of plasma modelled by

$$\partial_t f = \nabla \cdot \left(f \underbrace{\int_{\mathbb{R}^d} f_* |v - v_*|^{\gamma+2} \Pi[v - v_*](\nabla \log f - \nabla_* \log f_*) dv_*}_{\text{nonlocal nonlinear 'velocity'}} \right).$$

For $z \in \mathbb{R}^d$, the matrix $\Pi[z]$ is the projection onto $\{z\}^\perp$

$$\Pi[z] := \left(Id - \frac{z \otimes z}{|z|^2} \right).$$

$|v - v_*|^{\gamma+2}\Pi[v - v_*]$ can be **singular** for negative enough γ and it is also **non-negative semi-definite**.

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Landau as a parabolic equation

Repackage the **non-linear and non-local** (c.f. Gualdani and Guillen '16-'21) coefficients of Landau

$$\partial_t f = \nabla \cdot (\bar{a}[f]\nabla f - \bar{b}[f]f),$$

with

$$\bar{a}[f] = a * f = \int_{\mathbb{R}^d} |v - v_*|^{2+\gamma}\Pi[v - v_*]f(v_*)dv_* \gtrsim_f \langle v\rangle^\gamma,$$

and

$$\bar{b}[f] = \nabla\cdot\bar{a}[f] = -(d-1)\int_{\mathbb{R}^d} |v-v_*|^\gamma(v-v_*)f(v_*)dv_* < +\infty \impliedby f \in L^{\frac{d}{\gamma+1+d}}_{loc}.$$

When $\gamma = -d$, then $\nabla \cdot \bar{b}[f] = -C_d f$ which 'suggests' finite-time blow up from $\dot{f} = f^2$??? **OPEN** - community believes no blow up due to partial results [GGIV'19, DHJ'20, BGS'21].

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Landau as a stochastic process

As a diffusion equation, we consider solutions f which are probabilities. Suppose $V \sim f$. Then

$$dV = 2\bar{b}[f](V)dt + (2\bar{a}[f](V))^{\frac{1}{2}}\,dB_t.$$

This perspective of Landau has been studied by Fournier, Guérin, Hauray, Heydecker, and Mouhot. Good progress made for $\gamma \geq -2$ when $d = 3$. Partial progress for $\gamma \in (-3, -2)$ and degenerating as $\gamma \downarrow -3$.

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Mean-field limit derivation of Landau through interacting particle system? Consider evolution of N particles $V^1, \ldots, V^N$ which satisfy

$$dV^i = 2\bar{b}[\mu^N](V^i)dt + (2\bar{a}[\mu^N](V^i))^{\frac{1}{2}} dB_t^i, \quad i = 1, \ldots, N,$$

with $\mu^N = \frac{1}{N}\sum_{j=1}^N \delta_{V^j}$. Done by Fournier and Hauray '16 for $\gamma > -2$.

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The question of (global) **uniqueness** for the most physically relevant case $\gamma = -3$ is **OPEN**. Stochastic techniques (coupling method) used by Fournier and Guérin in 2009 to prove conditional uniqueness.

- $\gamma \in (-2, 0)$ - **global** uniqueness and stability conditional to finite initial moments and entropy.
- $\gamma \in (-3, -2]$ - **local** uniqueness and stability conditional on initial L^p, $p(\gamma) > 3/3 + \gamma$.

Outline

Mathematical challenges associated to different values of γ.

- $\gamma \in (0,1)$ corresponds to 'hard potentials' (Desvillettes and Villani [DV00, DV02]).
- $\gamma = 0$ corresponds to 'Maxwellian' molecules (Villani [V98]).
- $\gamma = -3$ corresponds to 'Coulomb' interactions (first global uniqueness result by Guo [G02] **using a perturbative argument**).

General uniqueness for $\gamma = -3$ is unknown, although Fournier and Guérin [FG09] have conditional results down to $\gamma > -3$ **using stochastic analytic techniques**. More recent work due to Fournier and Heydecker [FH21].

Regularity results by extending DeGiorgi-Nash-Moser in inhomogeneous case rely on **interplay between parabolic and kinetic theory** by Golse, Guérand, Imbert, Loher, Mouhot, Silvestre, Schwab, Vasseur [GIMV19,IM20,GH15,SS16,GM21, L22].

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More literature

- We want a **gradient flow description for the Landau equation** extending the ideas from Erbar and Villani [E16, V98].
- Erbar discovered the gradient flow characterisation of the **Boltzmann** equation with Maxwellian (bounded, $\gamma = 0$) kernels.
- The Landau equation can be derived from the Boltzmann equation through the **grazing collision limit** [DL92, D92].
- 'The lack of gradient flow structure contributes to the mathematical difficulty of the Boltzmann equation.' - Villani [V02].

H-theorem

$\mathcal{H}(f) = \int f \log f$ is a Lyapunov functional for Landau. Using $\log f$ as a test function gives

$$\mathcal{H}(f_t) - \mathcal{H}(f_0) = \\ - \int_0^t \underbrace{\frac{1}{2} \iint_{\mathbb{R}^{2d}} ff_* |v - v_*|^{\gamma+2} |\Pi[v - v_*](\nabla \log f - \nabla_* \log f_*)|^2 dv_* dv}_{=:D_L(f) \geq 0} \, dt.$$

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We interpret this entropy-dissipation equality as a **steepest gradient descent** of $\mathcal{H}$ under the flow of Landau - extend notion of H-solutions by Villani [V98].

$$\partial_t f = -\nabla_{d_L}\mathcal{H}(f), \quad \text{for some metric } d_L.$$

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Desvillettes proved the functional inequality [D15, D16]

$$\int \langle v \rangle^\gamma \frac{|\nabla f|^2}{f} + \int \langle v \rangle^\gamma \frac{|v \times \nabla f|^2}{f} \leq C(D_L(f) + 1),$$

$C > 0$ depends only on bounds for

$$\int \langle v \rangle^2 f, \quad \mathcal{H}(f).$$

Theorem (There is a 'good' metric d_L)

There exists a (pseudo)-metric d_L on the space of probability measures with bounded second moment, $\mathcal{P}_2(\mathbb{R}^d)$ such that

- *d_L metrises a complete topology*
- *d_L-convergent (resp. bounded) sequences are weakly convergent (resp. compact).*
- *$(\mu_0, \mu_1) \mapsto d_L(\mu_0, \mu_1)$ is weakly lower semicontinuous.*
- *d_L induces a geodesic space.*

We define gradient flows of $\mathcal{H}$ with respect to d_L to be curves f_t satisfying

$$\mathcal{H}(f_t) - \mathcal{H}(f_0) = -\frac{1}{2}\int_0^t |\dot{f}|^2_{d_L}(s)ds - \frac{1}{2}\int_0^t D_L(f_s)ds.$$

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Preview of results from CDDW20+ continued

Theorem (Landau Gradient flow for a regularised problem)

For a specific regularisation of the Landau equation [CHWW20], gradient flows of a **regularised** *Boltzmann entropy with respect to d_L* **exist** *and all gradient flows are* **equivalent** *to the usual notion of weak solutions in all dimensions $d \geq 3$ and $\gamma \in [-4, 0]$.*

For a regularisation parameter $\epsilon > 0$, these curves satisfy

$$\mathcal{H}^\epsilon(f_t) - \mathcal{H}^\epsilon(f_0) = -\frac{1}{2}\int_0^t |\dot{f}|^2_{d_L}(s)ds - \frac{1}{2}\int_0^t D_L^\epsilon(f_s)ds.$$

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Theorem (Landau gradient flow)

Fix $d = 3$ and $\gamma \in (-3, 0]$. All H-solutions [V98] f_t defined on $[0, T]$ subject to the **assumptions***;*

- $\mathcal{H}(f) \in L^\infty_t$,
- $D_L(f) \in L^1_t$,
- *there exists $p(\gamma) \in [1, \infty]$ such that*

$$(1 + |v|^2)^{1-\frac{\gamma}{2}} f_t(v) \in L^\infty_t(0, T; L^1_v \cap L^{p(\gamma)}_v(\mathbb{R}^3)),$$

are **equivalent** *to gradient flows of $\mathcal{H}$ with respect to d_L.*

The exponent $p(\gamma) \to \infty+$ as $\gamma \downarrow -3$.

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H-theorem for the heat equation

Heat equation may be written as gradient flow of $\mathcal{H}$ with respect to the 2-Wasserstein distance [JKO98]

$$\partial_t \rho = -\nabla_{W_2} \mathcal{H}(\rho) = \nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{H}}{\delta \rho} \right), \quad \frac{\delta \mathcal{H}}{\delta \rho} = \log \rho.$$

The entropy-dissipation of heat flow reads

$$\mathcal{H}(\rho_t) + \int_0^t \underbrace{\int_{\mathbb{R}^d} \rho |\nabla \log \rho|^2 dx}_{=: D_h(\rho)} \, dt = \mathcal{H}(\rho_0).$$

Dynamic Wasserstein distance

The Benamou-Brenier formulation [BB00] of 2-Wasserstein distance is

$$W_2^2(f_0, f_T) = \inf_{(f,M)} \left\{ T \int_0^T \int_{\mathbb{R}^d} \frac{|M|^2}{f} dxdt \;\middle|\; \begin{array}{c} \partial_t f + \nabla \cdot M = 0 \\ f(0) = f_0,\ f(T) = f_T \end{array} \right\}.$$

$M = -f \nabla \log f (= -\nabla f)$ is a candidate flux when f_t and $f_{t+\Delta t}$ are points along the heat semigroup

$$W_2^2(f_t, f_{t+\Delta t}) \leq \Delta t \int_t^{t+\Delta t} D_h(f_s) ds, \quad |\dot{f}|^2_{W_2}(t) \overset{!}{=} D_h(f_t).$$

Extend this idea for Landau; construct d_L and prove

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New ‘differential geometric’ structure

Recall

$$D_h(f) = \int_{\mathbb{R}^d} f|\nabla \log f|^2 dx$$

$$D_L(f) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} ff_*|v - v_*|^{2+\gamma}|\Pi[v - v_*](\nabla \log f - \nabla_* \log f_*)|^2 dv_* dv.$$

Define a differential operator for $\phi = \phi(v)$

$$[\tilde{\nabla}\phi](v, v_*) := |v - v_*|^{1+\frac{\gamma}{2}} \Pi[v - v_*](\nabla\phi - \nabla_*\phi_*),$$

so that

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Dynamic Landau distance

Recall

$$W_2^2(f_0, f_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|M|^2}{f} dxdt \;\middle|\; \begin{array}{c} \partial_t f + \nabla \cdot M = 0 \\ f(0) = f_0,\, f(1) = f_1 \end{array} \right\}.$$

We use now (the adjoint of) $\tilde{\nabla}$ in the actual definition

$$d_L^2(f_0, f_1) := \inf \left\{ \frac{1}{2} \int_0^1 \iint_{\mathbb{R}^{2d}} \frac{|M|^2}{ff_*} dv_* dvdt \;\middle|\; \begin{array}{c} \partial_t f + \frac{1}{2} \tilde{\nabla} \cdot M = 0 \\ f(0) = f_0,\, f(1) = f_1 \end{array} \right\}.$$

Now if f_0 and f_1 are evaluations of solutions to the Landau equation at $t = 0, 1$, then $M = -ff_* \tilde{\nabla} \log f$ is a candidate flux with

$$|\dot{f}|_{d_L}^2(t) \leq D_L(f_t).$$

Dynamic Landau distance

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$$W_2^2(f_0, f_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|M|^2}{f} dxdt \;\middle|\; \begin{array}{c} \partial_t f + \nabla \cdot M = 0 \\ f(0) = f_0,\, f(1) = f_1 \end{array} \right\}.$$

We use now (the adjoint of) $\tilde{\nabla}$ in the actual definition

$$d_L^2(f_0, f_1) := \inf \left\{ \frac{1}{2} \int_0^1 \iint_{\mathbb{R}^{2d}} \frac{|M|^2}{ff_*} dv_* dvdt \;\middle|\; \begin{array}{c} \partial_t f + \frac{1}{2} \tilde{\nabla} \cdot M = 0 \\ f(0) = f_0,\, f(1) = f_1 \end{array} \right\}.$$

Now if f_0 and f_1 are evaluations of solutions to the Landau equation at $t = 0, 1$, then $M = -ff_* \tilde{\nabla} \log f$ is a candidate flux with

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Apply arguments from Ambrosio, Gigli, Savaré and Dolbeault, Nazaret, Savaré [AGS08, DNS09]. The functional to be infimised (minimisers exist) in d_L is

$$\frac{1}{2}\int_0^1 \iint_{\mathbb{R}^{2d}} \frac{|M|^2}{ff_*} dv_* dv dt.$$

The integrand function takes the form

$$(x, y) \in \mathbb{R}^d \times (0, \infty) \mapsto \frac{1}{2}\frac{|x|^2}{y} \quad \textbf{is jointly convex.}$$

Calculus of variations + adaptation to gradient flows [DNS09] yield nice properties of d_L.

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Gradient flows $\iff$ H-solutions

Key ingredient is **chain rule** for $\partial_t f + \frac{1}{2}\tilde{\nabla} \cdot M = 0$ which is formally computed

$$\frac{d}{dt}\mathcal{H}(f_t) = \int_{\mathbb{R}^d} \partial_t f \log f = -\frac{1}{2}\int_{\mathbb{R}^d} \tilde{\nabla} \cdot M \log f = \frac{1}{2}\iint_{\mathbb{R}^{2d}} \tilde{\nabla} \log f \cdot M = \frac{1}{2}\iint_{\mathbb{R}^{2d}} \tilde{\nabla}\frac{\delta \mathcal{H}}{\delta f} \cdot M$$

H-solutions $\implies$ gradient flows: Set $M = -ff_*\tilde{\nabla}\log f$ so that

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Structure of regularised entropy

We consider, for $\epsilon > 0$,

$$\mathcal{H}^\epsilon(f) := \mathcal{H}(f * G^\epsilon) = \int [f * G^\epsilon] \log [f * G^\epsilon], \quad G^\epsilon(v) = C_{d,\epsilon} \exp(-|v|/\epsilon).$$

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Convergence of dissipations as $\epsilon \downarrow 0$

For a fixed f, we need to establish $D_L^\epsilon(f) \to D_L(f)$ given $D_L(f) < +\infty$, or in full detail

$$\iint_{\mathbb{R}^{2d}} ff_*|v-v_*|^{2+\gamma}|\Pi[v-v_*](\nabla G^\epsilon * \log[f*G^\epsilon](v) - \nabla G^\epsilon * \log[f*G^\epsilon](v_*))|^2$$

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Expand the square to effectively estimate two distinct terms

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Use of Extended Dominated Convergence

We do not have an integrable majorant directly, but we only care that the integral converges.

Idea: Suppose $I_\epsilon(v) = A(v)(B * G^\epsilon)(v)$ is a sequence and we want to show

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Commuting $G^\epsilon *$

Apply (many many many times) an extended Dominated Convergence Theorem after shuffling $G^\epsilon *$, for example

$$\int f_* |v_*|^2 \int f |v - v_*|^\gamma |\nabla G^\epsilon * \log[f * G^\epsilon](v)|^2 dv dv_* \leq \ldots$$
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Recognise Jensen's inequality

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Boundedness of Fisher information-type term from Desvillettes [D15, D16]

$$\int \langle v \rangle^\gamma \frac{|\nabla f|^2}{f} + \int \langle v \rangle^\gamma \frac{|v \times \nabla f|^2}{f} \leq C(D_L(f) + 1).$$

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Another motivation for regularisation

The d_L-gradient flow of $\mathcal{H}^\epsilon$ is the PDE $\partial_t f =$

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This PDE inherits conservation of mass, momentum, and energy, as well as a regularised H-theorem. Destroying the parabolicity allows for a numerical **particle method** as investigated in Carrillo, Craig, Patacchini '19 (and also recently by Craig et al.).

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Motivation from numerics (Carrillo, Hu, Wang, Wu, '20)

Landau is parabolic so the regularised entropy $\mathcal{H}^\epsilon$ is needed for particle solutions

$$f(t,v) \sim \mu^N(t,v) := \frac{1}{N}\sum_{i=1}^{N} \delta_{v^i(t)}(v).$$

Obtain an interacting particle system of ODEs for $i = 1, \ldots, N$

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We tested our method in 3D also for Coulomb molecules ($\gamma = -3$) and observed heuristic (almost) second order accuracy.

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Theorem (Existence and equivalence for $\epsilon > 0$ (CDDW20+))

Fix $\epsilon > 0$, an interval $[0, T]$, and $\gamma \in [-4, 0]$. Then there **exists** *a* **gradient flow** *of $\mathcal{H}^\epsilon$ with respect to d_L in $\mathcal{P}_2(\mathbb{R}^d)$. Furthermore, all weak solutions of the regularised Landau equation are* **equivalent** *to gradient flows.*

Proof.

Existence handled by JKO variational scheme [JKO98] given initial $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and time step $\tau > 0$

$$\nu_0^\tau := \mu_0, \quad \nu_n^\tau \in \mathrm{argmin}_{\lambda \in \mathcal{P}_2} \left[\mathcal{H}^\epsilon(\lambda) + \frac{1}{2\tau} {d_L}^2(\nu_{n-1}^\tau, \lambda) \right], \quad n \in \mathbb{N}.$$

Equivalence handled by chain rule

$$\frac{d}{dt}\mathcal{H}^\epsilon(f) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} \tilde{\nabla} \frac{\delta \mathcal{H}^\epsilon}{\delta f} \cdot M, \quad \text{given } \partial_t f + \frac{1}{2}\tilde{\nabla} \cdot M = 0.$$

□

Theorem (Existence and equivalence for $\epsilon > 0$ (CDDW20+))

Fix $\epsilon > 0$, an interval $[0, T]$, and $\gamma \in [-4, 0]$. Then there **exists** *a* **gradient flow** *of $\mathcal{H}^\epsilon$ with respect to d_L in $\mathcal{P}_2(\mathbb{R}^d)$. Furthermore, all weak solutions of the regularised Landau equation are* **equivalent** *to gradient flows.*

Proof.

Existence handled by JKO variational scheme [JKO98] given initial $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and time step $\tau > 0$

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Concluding remarks

- Provided a gradient flow description of Landau equation for soft potentials extending Erbar's treatment of Boltzmann for bounded potentials.
- Initiated gradient flow theory analysis of numerical experiments for ϵ-regularised problem.
- Introduced a new minimising movement procedure in the spirit of JKO for Landau.

Some future endeavours:

- Extend gradient flow formulation to $\gamma = -3$.
- Leverage gradient flow theory (e.g. generalised geodesic convexity) to prove uniqueness for $\gamma = -3$.
- Understand grazing collision limit in terms of gradient flows (e.g. using Γ-convergence). - Published in Nonlinear Analysis '22 ✓!
- Prove rigorous approximation of ϵ-regularised problem to Landau. - Convergence of particle method in preparation (for a fixed $\epsilon > 0$).

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