Homogenization of some quasi-linear elliptic equations with gradient constraints

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Outline

A linear model for elastic torsion

- Q is a cylindrical bar with periodic and identical cylindrical cavities
- $\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon^i$ is the cross-section of the the material
- $\varepsilon > 0$ is size of the period

According to Lanchon (1970), in the linear homogeneous isotropic case, the study of the elastic torsion of this bar leads to the following problem

$$
\begin{aligned}
-\Delta u_\varepsilon &= 2\mu\theta \qquad \text{in } \Omega_\varepsilon \\
u_\varepsilon &= \text{const} \qquad \text{on } \partial B_\varepsilon^i \\
u_\varepsilon &= 0 \qquad \text{on } \partial\Omega \\
\int_{\partial B_\varepsilon^i} \nabla u_\varepsilon \cdot \nu \, d\sigma &= 2\mu\theta |B_\varepsilon^i|
\end{aligned}
$$

where μ rigidity modulus, θ twist's angle and u_ε stress function, from which the stress tensor can be recovered. Here $|B_\varepsilon^i|$ denotes the area of the cross-section of each cavity, and ν is the exterior unit normal at the boundary of Ω_ε.
The study of the asymptotic behaviour of the solutions u_ε as $\varepsilon \to 0$ is due to Cioranescu-SaintJeanPaulin 1979

The same type of system appears for the electrostatic potential u_ε in presence of conducting inclusions B_ε^i

$$\begin{aligned} -\Delta u_\varepsilon &= g_\varepsilon && \text{in } \Omega_\varepsilon \\ u_\varepsilon &= \text{const} && \text{on } \partial B_\varepsilon^i \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \\ \int_{\partial B_\varepsilon^i} \nabla u \cdot \nu \, d\sigma &= \int_{B_\varepsilon^i} g_\varepsilon \, dx \end{aligned}$$

Here the boundary of the domain $\Omega \subset \mathbb{R}^3$ is grounded, $g_\varepsilon = 0$ in Ω_ε and $g_\varepsilon = 4\pi e/|B_\varepsilon^i|$ if each conductor B_ε^i has electric charge e. We refer, e. g., to J. Rauch and M. Taylor 1975, or to A. A. Kolpakov, A. G. Kolpakov 2009.

The above problems may be written in a variational form as:

find $u_\varepsilon \in H_0^1(\Omega)$ and $\nabla u_\varepsilon = 0$ in $B_\varepsilon = \cup_i B_\varepsilon^i$, such that

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx = \int_\Omega g_\varepsilon v \, dx$$

for all $v \in H_0^1(\Omega)$ with $\nabla v = 0$ in $B_\varepsilon = \cup_i B_\varepsilon^i$.

Several generalizations are possible. In this talk, we replace ∇u_ε with a vector of the form $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$ that takes into account space oscillations and non linear dependence on the gradient ∇u_ε.

We point out that several related results are available for related minimum problems of the type

$$\min\left\{\int_\Omega \left(f\left(\frac{x}{\varepsilon}, \nabla u\right) - 2gu\right)\, dx : u \in V^\varepsilon\right\} \qquad (1)$$

where $V^\varepsilon \subset H_0^1(\Omega)$are obtained in the framework of Γ-convergence theory.

We mention in particular

- A. Braides, A. Garroni 1995, about non-linear elastic materials with stiff and soft inclusions,
- L. Carbone, R. De Arcangelis, U. De Maio 2000, for the homogenization of media with periodically distributed conductors, and
- R. De Arcangelis, A. Gaudiello, G. Paderni 1996, for more general constrained variational problems.

The problem set up

From now on, we consider the following variational equation for $u_\varepsilon \in K^\varepsilon$:

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla\varphi \, dx = \int_\Omega g\varphi \, dx, \quad \forall \varphi \in K^\varepsilon \tag{2}$$

where $g \in L^2(\Omega)$ is **independent of** ε and

$$K^\varepsilon = \left\{ v \in H_0^1(\Omega) : \nabla v(x) = 0 \text{ a.e. in } \varepsilon B \cap \Omega \right\} \tag{3}$$

Let $Y = (0,1)^n$ denote the periodicity cell, $B \subset \mathbb{R}^n$ be the closure of a Lipschitz Y-periodic open set. We assume that B is **disperse,** in the sense that $B \cap Y \subset\subset Y$. We also assume that $B \cap Y$ has a **finite** number of connected components.
We denote the complement of the inclusions $\varepsilon B \cap \Omega$ by Ω_ε.

Assumptions on $a(y, \xi)$

The function $a = a(y, \xi) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is measurable and Y-periodic in $y \in \mathbb{R}^n$ for every $\xi \in \mathbb{R}^n$ and satisfies monotonicity and regularity conditions:
$\exists\, \alpha, L > 0$ such that

$$(a(y, \xi_1) - a(y, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^2, \tag{4}$$

$$|a(y, \xi_1) - a(y, \xi_2)| \leqslant L |\xi_1 - \xi_2|, \tag{5}$$

$$a(y, 0) = 0 \quad \text{for a.e. } y \in \mathbb{R}^n. \tag{6}$$

for a.e. $y \in \mathbb{R}^n, \forall\, \xi_1, \xi_2 \in \mathbb{R}^n$.

Remark

Notice that if g is replaced by

$$g_\varepsilon = \frac{g}{|Y \cap B|} \chi_{\varepsilon B}$$

where $\chi_{\varepsilon B}$ represents the characteristic function of the inclusions εB, **the asymptotic problem does not change**.

In fact, if g is replaced by $g_\varepsilon = h\chi_{\varepsilon B}$ with $h \in L^2(\Omega)$, and we compare the behaviour of u_ε and v_ε, the solutions of (2) corresponding to g and g_ε respectively, by the strict monotonicity of $a(y,\cdot)$ it follows that

$$\begin{aligned}\alpha \int_\Omega |\nabla u_\varepsilon - \nabla v_\varepsilon|^2\,dx &\leqslant \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\right](\nabla u_\varepsilon - \nabla v_\varepsilon)\,dx \\ &= \int_\Omega (g - h\chi_{\varepsilon B})(u_\varepsilon - v_\varepsilon)\,dx\end{aligned}$$

Now, if $u_\varepsilon, v_\varepsilon \to u, v$ respectively, then

$$\int_\Omega (g - h\chi_{\mathbb{R}^n \cap \varepsilon B})(u_\varepsilon - v_\varepsilon)\,dx \xrightarrow{\varepsilon \to 0} \int_\Omega (g - h|Y \cap B|)(u - v)\,dx,$$

where $|Y \cap B|$ denotes the Lebesgue measure of $Y \cap B$. If $h = \frac{g}{|Y \cap B|}$ this yields $u = v$, which means that the asymptotic behaviour of u_ε is the same as the one of v_ε.

Preliminary results

- Existence and uniqueness of the solution u_ε
- a-priori estimates for the solutions u_ε and the momenta $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$
- Compactness
- The cell problem
- The homogenized symbol/operator

Existence and a-priori estimates

Theorem

Under the above assumptions, for every $g \in L^2(\Omega)$, problem (2) has exactly one solution $u_\varepsilon \in K^\varepsilon$. Moreover,

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c, \tag{7}$$

$$\left\| a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \right\|_{L^2(\Omega)^n} \leq Lc \tag{8}$$

where $c = \alpha^{-1} c_P \|g\|_{L^2(\Omega)}$ is independent of ε, and c_P denotes the constant for the Poincaré inequality in $H_0^1(\Omega)$.

The proof relies on standard properties of monotone operators

Compactness and questions

From the a priori estimates (7), (8) and by Rellich's theorem we have, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega), \tag{9}$$

$$b_\varepsilon(x) =: a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \rightharpoonup \hat{a} \quad \text{in } L^2(\Omega)^n, \tag{10}$$

and it is natural to ask:

- How the limits u and $\hat{a}$ are related ?
- Do we have $\hat{a} = a_{\mathrm{hom}}(\nabla u)$?
- May we find a limit (homogenized) problem of the type

$$-\mathrm{div} a_{\mathrm{hom}}(\nabla u) = g?$$

One difficulty and one advantage

- To pass to the limit as $\varepsilon \to 0$ in

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla\varphi \, dx = \int_\Omega g\varphi \, dx, \quad \forall\varphi \in K^\varepsilon$$

is not straightforward, since the test functions

$$\varphi \in K^\varepsilon = \left\{ v \in H_0^1(\Omega) : \nabla v(x) = 0 \text{ a.e. in } \varepsilon B \cap \Omega \right\}$$

depend on ε.

- Taking the test functions $\varphi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$

$$-\operatorname{div} b_\varepsilon(x) = g \quad \text{in } \mathcal{D}'(\Omega_\varepsilon) \text{ and in } L^2(\Omega_\varepsilon)$$

- We are allowed to **modify** $b_\varepsilon = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$ **inside the inclusions** $\varepsilon B \cap \Omega$, where $\nabla\varphi(x) = 0$.

Which cell problem ?

In order to determine such cell problem, we have taken into account the homogenization of minimum problems of the type

$$\min \left\{ \int_{\Omega} \left(|\nabla u|^2 - 2gu \right) \, dx : u \in V^{\varepsilon} \right\},$$

considered by G. Cardone, A. Corbo Esposito, G.A. Yosifian, V.V. Zhikov, 2004, for a quite general convex set $V^{\varepsilon} \subset H_0^1(\Omega)$.

When $V^{\varepsilon} = K^{\varepsilon}$, then our equation with $a(y, \xi) = \xi$ is the Euler-Lagrange equation of the above minimum problem.

The results of [CCYZ] suggest then to choose the Euler-Lagrange equation of the cell problem corresponding to that minimum problem as a "good candidate" for the cell problem in our case.

The cell problem

From now on, we denote by $H^1_\sharp(Y)$ the subspace of $H^1_{\rm loc}(\mathbb{R}^n)$ of functions v that are Y-periodic and have mean-value zero in the periodicity cell Y, equipped with the norm $||v||_{H^1_\sharp(Y)} = ||\nabla v||_{L^2(Y)}$. For every given $\xi \in \mathbb{R}^n$, we consider the following closed convex subset of $H^1_\sharp(Y)$

$$K_\xi = \left\{ v \in H^1_\sharp(Y) : \xi + \nabla v(y) = 0 \ \ \text{a.e.in} \ \ B \right\}, \quad \xi \in \mathbb{R}^n.$$

In particular, for $\xi = 0$, K_0 is a closed subspace of $H^1_\sharp(Y)$. In view of the above considerations, we formulate the following cell problem in weak form

$$\begin{cases} \displaystyle\int_Y a(y, \xi + \nabla w_\xi) \cdot \nabla \varphi \, dy = 0, \quad \forall \varphi \in K_0 \\ w_\xi \in K_\xi. \end{cases} \tag{11}$$

Properties: existence, apriori-estimates, continuity in ξ

We prove that:

- for any fixed $\xi \in \mathbf{R}^n$, there exists unique solution w_ξ of the cell problem
- at each connected component Γ of the boundary $\partial B \cap Y$, the solution w_ξ satisfies

$$\int_\Gamma a(y, \xi + \nabla w_\xi) \cdot \nu_B \, d\sigma = 0 \tag{12}$$

- there is a constant $c > 0$ such that

$$\|\xi + \nabla w_\xi\|_{L^2(Y)} \leqslant c|\xi|, \quad \forall \xi \in \mathbb{R}^n$$

The homogenized symbol

Let us define $a_{\text{hom}} = a_{\text{hom}}(\xi, \eta) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
as

$$a_{\text{hom}}(\xi, \eta) = \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot (\eta + \nabla w_\eta)\, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n,$$

where $w_\xi \in K_\xi$ and $w_\eta \in K_\eta$ are solutions of the cell problem.

The homogenized symbol

We prove that there exists $a_0 : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(a_0(\xi_1) - a_0(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^2, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n$$

$$|a_0(\xi_1) - a_0(\xi_2)| \leqslant L' |\xi_1 - \xi_2|,$$

$$a_0(0) = 0$$

and

$$a_0(\xi) \cdot \eta = a_{\text{hom}}(\xi, \eta) \quad \forall \xi, \eta \in \mathbb{R}^n$$

Main result

Theorem

Let u_ε be the unique solution of the equation (2). Then $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, where u is the unique solution of the homogenized equation

$$\int_\Omega a_0(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega g \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega) \qquad (13)$$

with $a_0(\xi) \cdot \eta = a_{\text{hom}}(\xi, \eta)$ for all $\xi, \eta \in \mathbb{R}^n$.

Main tools in the proof

- extension operator
- oscillating test functions
- properties of a_0
- compensated compactness

The extension lemma (Cioranescu-SaintJeanPaulin 1979)

Lemma

Let $z \in L^2(Y \setminus B)^n$ *and* $g \in L^2(Y)$ *such that*

$$-div\, z = g \qquad in\ \mathcal{D}'(Y \setminus B), \tag{14}$$

$$\int_{Y\setminus B} z \cdot \nabla\varphi\, dy = \int_Y g\varphi\, dy \quad \forall\, \varphi \in C_0^\infty(Y) :\ \nabla\varphi|_B = 0, \tag{15}$$

then there exists $\tilde{z} \in L^2(Y)^n$ *such that*

$$-div\, \tilde{z} = g \qquad in\ Y\ and\ in\ \mathcal{D}'(Y), \tag{16}$$

$$\tilde{z} = z \qquad in\ Y \setminus B, \tag{17}$$

$$z \cdot \nu_B = \tilde{z} \cdot \nu_B \quad in\ Y \cap \partial B, \tag{18}$$

$$\int_{B\cap Y} |\tilde{z}|^2\, dy \leqslant c \left(\int_Y |g|^2\, dy + \int_{Y\setminus B} |z|^2\, dy \right). \tag{19}$$

where ν_B *denotes the unit normal vector to the boundary of* B*, and* c *is a constant independent of* z *and* g*.*

Extension of $b_\varepsilon = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$

We modify the momenta

$$b_\varepsilon(x) = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) \tag{20}$$

over the sets εB.

In fact, using the Lemma, for any $\Omega' \subset\subset \Omega$, $\varepsilon < \varepsilon_0(\Omega')$, there exists an extension $\tilde{b}_\varepsilon \in L^2(\Omega')^n$ of $b_\varepsilon|_{\Omega_\varepsilon}$ such that

$$-\mathrm{div}_x \tilde{b}_\varepsilon(x) = g(x) \quad \text{in } \mathcal{D}'(\Omega'), \tag{21}$$

$$\tilde{b}_\varepsilon = b_\varepsilon \quad \text{in } \Omega' \setminus \varepsilon B \tag{22}$$

$$\int_{\Omega'} |\tilde{b}_\varepsilon(x)|^2\, dx \leqslant c\left(\int_\Omega |\varepsilon g(x)|^2\, dx + \int_{\Omega\setminus\varepsilon B} |b_\varepsilon(x)|^2\, dx\right). \tag{23}$$

Repeating the construction for an increasing sequence of open subsets $\Omega'_j \subset\subset \Omega$ such that $\cup_j \Omega'_j = \Omega$, we can prove that

there exists $b \in L^2_{\mathrm{loc}}(\Omega)^n$ and there exists a subsequence of $\varepsilon \to 0$ (not relabeled), such that for all $j \geq 1$

$$\tilde{b}^{(j)}_\varepsilon \rightharpoonup b \qquad \text{weakly in } L^2(\Omega'_j)^n,$$

$$-\mathrm{div}_x \tilde{b}^{(j)}_\varepsilon(x) = g(x) = -\mathrm{div}_x b \quad \text{in } \mathcal{D}'(\Omega'_j)$$

Construction of a_0

Let us define by $\beta = \beta(y, \xi)$ the function

$$\beta(y, \xi) = a\left(y, \xi + \nabla w_\xi(y)\right). \tag{24}$$

For any $\xi \in \mathbb{R}^n$, the function $\beta(\cdot, \xi) \in [L^2_{\text{loc}}(\mathbb{R}^n)]^n$, it is Y-periodic, and has the following properties:

$$-\text{div}_y\, \beta(y, \xi) = 0 \quad \text{in } \mathcal{D}'(Y \setminus B), \tag{25}$$

$$\int_{Y \setminus B} \beta(y, \xi) \cdot \nabla\varphi\, dy = 0, \quad \forall\, \varphi \in \mathcal{D}'(Y \setminus B) : \ \nabla\varphi|_B = 0. \tag{26}$$

Construction of a_0

By the Extension Lemma (with $g = 0$) there exists an extension

$$\tilde{\beta} = \tilde{\beta}(\cdot, \xi) \in L^2(Y)^n \tag{27}$$

such that

$$- \operatorname{div} \tilde{\beta}(y, \xi) = 0 \quad \text{in } Y, \text{ in } \mathcal{D}'(Y), \tag{28}$$

$$\tilde{\beta} = \beta \quad \text{in } Y \setminus B, \tag{29}$$

$$\int_B |\tilde{\beta}|^2 \, dx \leqslant c \int_{Y \setminus B} |\beta|^2 \, dx, \tag{30}$$

with c independent of β.

Construction of a_0

Let us define

$$\tilde{\beta}_\varepsilon(x) = \tilde{\beta}\left(\frac{x}{\varepsilon}\right). \tag{31}$$

The εY-periodic function $\tilde{\beta}_\varepsilon$ has the following properties

$$-\operatorname{div}\tilde{\beta}_\varepsilon = 0 \text{ in } \mathbb{R}^n, \tag{32}$$

$$\tilde{\beta}_\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right) \text{ in } \mathbb{R}^n \setminus \varepsilon B, \tag{33}$$

and

$$\tilde{\beta}_\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y \tilde{\beta}(y,\xi)\,dy \quad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^n) \tag{34}$$

We set

$$a_0(\xi) = \int_Y \tilde{\beta}(y,\xi)\,dy \tag{35}$$

Passage to the limit

Given $\xi \in \mathbb{R}^n$, let us take the solution w_ξ of the cell problem and set

$$v_\varepsilon(x) = \varepsilon\, w_\xi\left(\frac{x}{\varepsilon}\right) + \xi \cdot x. \tag{36}$$

By the mean value property, we have

$$v_\varepsilon \to \xi \cdot x \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^n), \tag{37}$$

$$\nabla v_\varepsilon = \nabla_y w_\xi + \xi \rightharpoonup \xi \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^n), \tag{38}$$

as $\varepsilon \to 0$. Moreover,

$$a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x)\right) = \beta_\varepsilon(x)$$

Passage to the limit

By the monotonicity of $a(y,\cdot)$, for any $\varphi \in D(\Omega)$, $\varphi \geq 0$, we have

$$\int_\Omega (b_\varepsilon - \beta_\varepsilon)\cdot(\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x)\,dx =$$

$$= \int_\Omega \left(a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x)\right)\right)\cdot(\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x)\,dx \geqslant 0.$$

Since $\nabla u_\varepsilon - \nabla v_\varepsilon = -(\xi + \nabla w_\xi(y)) = 0$ in $\varepsilon B \cap \Omega$, we are allowed to modify $b_\varepsilon, \beta_\varepsilon$ in the inclusions.

Then considering the extensions $\tilde{b}_\varepsilon^{(j)}(x)$ of $b_\varepsilon(x)$ defined in $\Omega' = \Omega'_j$ and the periodic extension $\tilde{\beta}_\varepsilon(x)$ of $\beta(\frac{x}{\varepsilon}) = a(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x))$ from the perforated set to $\mathbb{R}^n$ the above inequality can be cast as

$$\int_{\Omega'_j} \left(\tilde{b}_\varepsilon^{(j)}(x) - \tilde{\beta}_\varepsilon(x)\right)\cdot(\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x)\,dx \geqslant 0.$$

Now, since $-\mathrm{div}\tilde{b}^{(j)} = g$ and $\mathrm{div}\tilde{\beta}_\varepsilon = 0$, we can pass to the limit in the inequality by compensated compactness:

$$\int_{\Omega_j'} \left(\tilde{b}_\varepsilon^{(j)}(x) - \tilde{\beta}_\varepsilon(x)\right) \cdot (\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x)\,dx \geqslant 0$$

and we get

$$\int_\Omega (b(x) - a_0(\xi)) \cdot (\nabla u(x) - \xi)\varphi(x)\,dx \geq 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$

$$\Downarrow$$

$$(b(x) - a_0(\xi) \cdot (\nabla u(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$.

By the continuity of a_0 this yields that

$$b(x) = a_0(\nabla u(x)) \quad \text{a.e. in } \Omega$$

Conclusion

- Since $-\mathrm{div} b = g$, and a_0 is strictly monotone, we can conclude that the whole sequence u_ε tends to the unique solution u of the homogenized equation

$$-\mathrm{div} a_0(\nabla u) = g$$

- Moreover, since $a_0(\xi) \cdot \eta = a_{\mathrm{hom}}(\xi, \eta)$, the result does not depend on the extension operator.

Thank you for your attention