Homogenization of a nonlinear monotone problem with nonlinear Signorini boundary conditions in a domain with highly rough boundary

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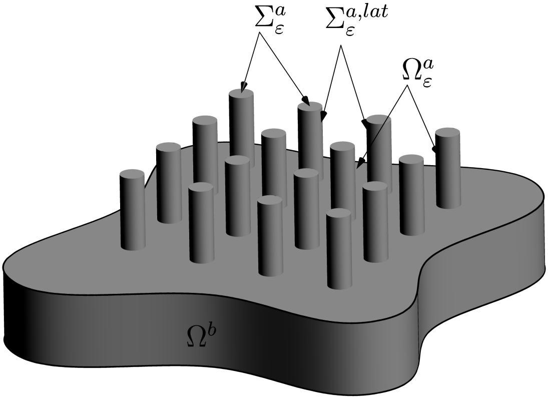
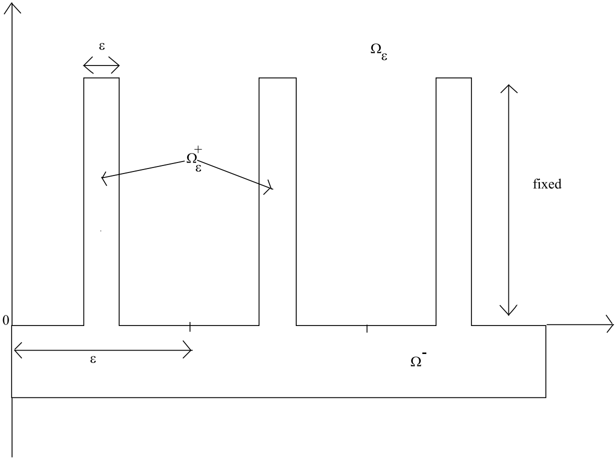
Domain Ω_ε with highly rough boundary





ε
Ω_ε
Ω_ε^+
fixed
0
ε
Ω^-

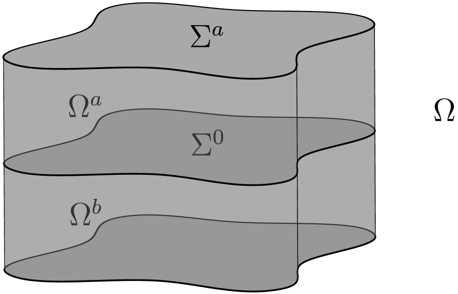
$$
\begin{cases}
\Omega^a_\varepsilon = \displaystyle\bigcup_{\{k\in\mathbb{Z}^{N-1}:\, \varepsilon\omega+\varepsilon k\subset\subset\Omega'\}} (\varepsilon\omega+\varepsilon k)\times[0,l^a[, \\
\Sigma^a_\varepsilon = \displaystyle\bigcup_{\{k\in\mathbb{Z}^{N-1}:\, \varepsilon\omega+\varepsilon k\subset\subset\Omega'\}} (\varepsilon\omega+\varepsilon k)\times\{l^a\}, \\
\Sigma^{a,lat}_\varepsilon = \displaystyle\bigcup_{\{k\in\mathbb{Z}^{N-1}:\, \varepsilon\omega+\varepsilon k\subset\subset\Omega'\}} (\varepsilon\partial\omega+\varepsilon k)\times]0,l^a[, \\
\Omega^b = \Omega'\times]-l^b,0[, \quad \Omega_\varepsilon = \Omega^a_\varepsilon\cup\Omega^b, \\
\Omega^a = \Omega'\times]0,l^a[, \quad \Omega = \Omega'\times]-l^b,l^a[, \\
\Sigma^0 = \Omega'\times\{0\}, \quad \Sigma^a = \Omega'\times\{l^a\},
\end{cases}
$$

where $N\in\mathbb{N}\setminus\{1\}$,

$$\omega\subset\subset]0,1[^{N-1} \text{ and } \Omega'\subset\mathbb{R}^{N-1}$$

are two connected bounded open sets with Lipschitz boundary, l^a, $l^b\in]0,+\infty[$, and

$$\{\varepsilon\}\subset]0,1[$$



$\chi_{\Omega_\varepsilon^a} \rightharpoonup |\omega'|$ weakly-star in $L^\infty(\Omega^a)$, as $\varepsilon \to 0$.

Some motivations

• The denticles on the skin of the shark create tiny vortices that reduce drag to make swimming more efficient. They also allow the shark to swim silently.

Similarly, racing boats present denticles under the hydrofoil hull and racing cars have serrated spoiler.

• Sensors used in automotive applications (ABS, ESP, Airbags, etc.) have a comb-shape. The acceleration component into the direction of the motion activates the device, deforming the teeth.

• Air flow through compression system in turbo machine such as a jet engine.

• Bridges on pillars, frameworks of houses, etc.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The impossibility to approach a variational problem in such a domain directly with numerical methods, due to a large number of mesh points required by the rough boundary of Ω_ε, suggests to develop an asymptotic analysis of the problem, as ε vanishes.

Then, the goal is to approach the problem, when ε gets smaller, with a model in Ω which can be numerically solved.

Model problem with homogeneous Dirichlet boundary condition

Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique weak solution to the following problem

$$\begin{cases} -\Delta u_\varepsilon = f, \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0, \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

where $f \in L^2(\Omega)$.

By now on, $\widetilde{v}$ denotes the zero-extension to Ω of any function v defined in Ω_ε.

Theorem

$$\widetilde{u_\varepsilon} \to 0 \textit{ strongly in } H^1(\Omega^a), \quad u_\varepsilon \to u^b \textit{ strongly in } H^1(\Omega^b),$$

as ε tends to zero, where u^b is the unique weak solution to the following problem

$$\begin{cases} -\Delta u^b = f, \textit{ in } \Omega^b, \\ u = 0, \textit{ on } \partial\Omega^b. \end{cases}$$

Sketch of the proof

$$\exists c > 0 : \quad \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c, \quad \forall \varepsilon.$$

Consequently, up to a subsequence,

$$\widetilde{u_\varepsilon} \rightharpoonup v \text{ weakly in } H^1(\Omega),$$

as ε tends to zero. Then, passing to the limit in

$$\widetilde{u_\varepsilon} = \chi_{\Omega^a_\varepsilon} \widetilde{u_\varepsilon}, \text{ in } \Omega^a, \quad \forall \varepsilon,$$

one obtains that

$$v = |\omega'| v, \text{ a.e. in } \Omega^a,$$

i.e.

$$v = 0, \text{ a.e. in } \Omega^a.$$

Model problem with homogeneous Neumann boundary condition

Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique weak solution to the following problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f, \text{ in } \Omega_\varepsilon, \\ \\ Du_\varepsilon \cdot \nu_\varepsilon = 0, \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

where $f \in L^2(\Omega)$ and ν_ε denotes the unit outer normal on $\partial\Omega_\varepsilon$.

By now on, $\widetilde{v}$ denotes the zero-extension to Ω of any function v defined in Ω_ε.

By using the method of oscillating test functions introduced by *L. Tartar (Collège de France - 1977)*, *R. Brizzi and J. P. Chalot (PhD Thesis - Nice University - 1978)* proved the following result.

Theorem

$$u_\varepsilon \rightharpoonup u^b \text{ weakly in } H^1(\Omega^b), \quad \widetilde{u_\varepsilon} \rightharpoonup |\omega| u^a \text{ weakly in } L^2(\Omega^a),$$

$$\widetilde{Du_\varepsilon} \rightharpoonup \left(0, \cdots, 0, |\omega| \frac{\partial u^a}{\partial x_N}\right) \text{ weakly in } (L^2(\Omega^a))^N,$$

as ε *tends to zero, where* $u = (u^a, u^b) \in V^2(\Omega) =$

$$\left\{ v = (v^a, v^b) \in L^2(\Omega^a) \times H^1(\Omega^b),\ \frac{\partial v^a}{\partial x_N} \in L^2(\Omega^a),\ v^a = v^b \text{ on } \Sigma^0 \right\}$$

is the unique weak solution to the following problem

$$\begin{cases} -\dfrac{\partial^2 u^a}{\partial x_N^2} + u^a = f,\ \text{in } \Omega^a, \\[2ex] -\Delta u^b + u^b = f,\ \text{in } \Omega^b, \\[2ex] u^a = u^b,\ \ |\omega| \dfrac{\partial u^a}{\partial x_N} = \dfrac{\partial u^b}{\partial x_N},\ \text{on } \Sigma^0, \\ \dfrac{\partial u^a}{\partial x_N} = 0,\ \text{on } \Sigma^a, \quad Du^b \cdot \nu = 0,\ \text{on } \partial\Omega^b \setminus \Sigma^0. \end{cases}$$

The method of oscillating test functions of L. Tartar

Sketch of the proof when $N = 2$

Choosing $v = u_\varepsilon$ as test function in the problem implies

$$\exists c > 0 : \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} < c \quad \forall \varepsilon.$$

Consequently, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and

$$u = (u^a, u^b) \in V^2(\Omega), \quad d' \in L^2(\Omega^a),$$

such that, as ε tends to zero,

$$\widetilde{u^a_\varepsilon} \rightharpoonup \theta u^a, \quad \widetilde{D_{x_2} u^a_\varepsilon} = D_{x_2} \widetilde{u^a_\varepsilon} \rightharpoonup \theta D_{x_2} u^a \quad \text{weakly in } L^2(\Omega^a),$$

$$\widetilde{D_{x_1} u^a_\varepsilon} \rightharpoonup \theta d' \text{ weakly in } L^2(\Omega^a),$$

$$u^b_\varepsilon \rightharpoonup u^b \text{ weakly in } H^1(\Omega^b).$$

To identify d', one uses the method of oscillating test functions introduced by L. Tartar.
Let $\tau \in W_0^{1,\infty}(]0,1[)$ be a function such that

$$\tau(x_1) = x_1, \text{ in } \omega.$$

Let $\widehat{\tau}$ be the $[0,1]$-periodic extension to $\mathbb{R}$ of a τ. Then, the sequence $\{w_\varepsilon\}_\varepsilon$ defined by

$$w_\varepsilon : x = (x_1, x_2) \in \Omega^a \to \varepsilon \widehat{\tau}\left(\frac{x_1}{\varepsilon}\right)$$

satisfies the following properties:

$$Dw_\varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ in } \Omega_\varepsilon, \quad \forall \varepsilon,$$

$$w_\varepsilon \to 0 \text{ strongly in } L^\infty(\Omega), \text{ as } \varepsilon \to 0.$$

Choosing $v = w_\varepsilon \varphi$ with $\varphi \in C_c^\infty(\Omega^a)$ as test function in the problem gives

$$\int_{\Omega^a} \widetilde{Du_\varepsilon^a} D\varphi \, w_\varepsilon dx + \int_{\Omega^a} \widetilde{Du_\varepsilon^a} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi dx + \int_{\Omega^a} \widetilde{u_\varepsilon^a} \varphi w_\varepsilon dx$$
$$= \int_{\Omega^a} \chi_{\Omega_\varepsilon^a} f \varphi w_\varepsilon dx, \quad \forall \varphi \in C_c^\infty(\Omega^a), \quad \forall \varepsilon.$$

Passing to the limit, as ε tends to zero, gives

$$\int_{\Omega^a} \begin{pmatrix} \theta d' \\ \theta D_{x_2} u^a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega^a)$$

which provides

$$d' = 0, \text{ a.e. in } \Omega^a.$$

Remark

It is not restrictive to study the asymptotic behavior of $\widetilde{u_\varepsilon}$ in Ω^a. In fact, if $P_\varepsilon \in \mathcal{L}(H^1(\Omega_\varepsilon^a), H^1(\Omega^a))$ is an extension operator such that

$$P_\varepsilon u_\varepsilon \rightharpoonup w \text{ weakly in } H^1(\Omega^a),$$

passing to the limit in

$$\widetilde{u_\varepsilon} = \chi_{\Omega_\varepsilon^a} P_\varepsilon u_\varepsilon, \text{ in } \Omega^a,$$

one obtains that

$$u^a = w \text{ in } \Omega^a.$$

Counterexample (M. Zerner)

In general, it is not possible to build a bounded sequence $\{P_\varepsilon\}_\varepsilon$ of extension operators $P_\varepsilon \in \mathcal{L}(H^1(\Omega^a_\varepsilon), H^1(\Omega^a))$. In fact, let

$$\Omega^a_\varepsilon = \bigcup_{\left\{k\in\mathbb{N}:\, \varepsilon\right]0,\frac{1}{2}\left[+\varepsilon(k-1)\subset\subset\right]0,1\left[\right\}} \left(\left(\varepsilon\left]0,\frac{1}{2}\right[+\varepsilon(k-1)\right)\times]0,l^a[\right),$$

and

$$v_\varepsilon = \begin{cases} v \text{ in } \left(\varepsilon\left]0,\frac{1}{2}\right[+\varepsilon(k-1)\right)\times]0,l^a[, \text{ if } k \text{ is odd,} \\ -v \text{ in } \left(\varepsilon\left]0,\frac{1}{2}\right[+\varepsilon(k-1)\right)\times]0,l^a[, \text{ if } k \text{ is even,} \end{cases}$$

where $v \in C^1_0]0,l^a[$. Then,

$$\exists c>0 : \|v_\varepsilon\|_{H^1(\Omega^a_\varepsilon)} \le c \quad \forall\varepsilon,$$

while, if $P_\varepsilon \in \mathcal{L}(H^1(\Omega^a_\varepsilon), H^1(\Omega^a))$ is an extension operator, an easy computation shows that

$$\lim_{\varepsilon\to 0}\left\|\frac{\partial P_\varepsilon v_\varepsilon}{\partial x_1}\right\|_{H^1(\Omega^a)} = +\infty.$$

Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique weak solution to the following problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f, \text{ in } \Omega_\varepsilon, \\ \\ Du_\varepsilon \cdot \nu = 0, \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

with $N = 2$ and $f \in H^1(\Omega)$. Then, *R. Brizzi and J. P. Chalot (1978)* proved the existence of a sequence $\{P_\varepsilon\}_\varepsilon$ of extension operators $P_\varepsilon \in \mathcal{L}(H^1(\Omega_\varepsilon^a), H^1(\Omega^a))$ and a constant $c > 0$ such that

$$\|P_\varepsilon u_\varepsilon\|_{H^1(\Omega^a)} \leq c \quad \forall \varepsilon.$$

If N=3, *T. Mel'nyk (Z. Anal. Anwendungen - 1999)* proved the same result if the functions f have not "strong scattering of the values on the neighboring cylinders".

Non homogeneous Neumann boundary conditions

Let $\lambda \in [0, +\infty[$ and $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique weak solution to the following problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \\ Du_\varepsilon \cdot \nu_\varepsilon = \gamma \varepsilon^\lambda \text{ on } \Sigma_\varepsilon^{a,\text{lat}}, \\ \\ Du_\varepsilon \cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Sigma_\varepsilon^{a,\text{lat}}, \end{cases}$$

where $f \in L^2(\Omega)$ and $\gamma \in \mathbb{R}$.

A.G. (Ricerche Mat. - 1994) proved the following result.

Theorem

If $\lambda \geq 1$, then

$$u_\varepsilon \rightharpoonup u^b \text{ weakly in } H^1(\Omega^b), \quad \widetilde{u_\varepsilon} \rightharpoonup |\omega| u^a \text{ weakly in } L^2(\Omega^a),$$

$$\widetilde{Du_\varepsilon} \rightharpoonup \left(0, \cdots, 0, |\omega| \frac{\partial u^a}{\partial x_N}\right) \text{ weakly in } (L^2(\Omega^a))^N,$$

as ε tends to zero, where $u = (u^a, u^b) \in V^2(\Omega)$ is the weak solution to the following problem

$$\begin{cases} -\dfrac{\partial^2 u^a}{\partial x_N^2} + u^a = \gamma \delta_{\lambda,1} \frac{|\partial\omega|}{|\omega|} + f^a \text{ in } \Omega^a, \\ -\Delta u^b + u^b = f^b \text{ in } \Omega^b, \\ u^a = u^b, \quad |\omega| \dfrac{\partial u^a}{\partial x_N} = \dfrac{\partial u^b}{\partial x_N} \text{ on } \Sigma^0, \\ \dfrac{\partial u^a}{\partial x_N} = 0 \text{ on } \Sigma^a, \\ Du^b \cdot \nu = 0 \text{ on } \partial\Omega^b \setminus \Sigma^0. \end{cases}$$

If $0 \leq \lambda < 1$, *then*

$$\exists \mu_1,\, \mu_2 \in]0,+\infty[\, : \ \frac{\mu_1}{\varepsilon^{1-\lambda}} \leq \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \frac{\mu_2}{\varepsilon^{1-\lambda}}, \quad \forall \varepsilon,$$

and the limit of $\varepsilon^{1-\lambda} u_\varepsilon$ *is*

$$\begin{cases} -\dfrac{\partial^2 u^a}{\partial x_N^2} + u^a = \gamma \frac{|\partial\omega|}{|\omega|} \ \textit{in } \Omega^a, \\[2ex] -\Delta u^b + u^b = 0 \ \textit{in } \Omega^b, \\[2ex] u^a = u^b, \ \ |\omega| \dfrac{\partial u^a}{\partial x_N} = \dfrac{\partial u^b}{\partial x_N} \ \textit{on } \Sigma^0, \\[2ex] \dfrac{\partial u^a}{\partial x_N} = 0 \ \textit{on } \Sigma^a, \\[2ex] Du^b \cdot \nu = 0 \ \textit{on } \partial\Omega^b \setminus \Sigma^0. \end{cases}$$

The matrix A and the coefficient a_0

Let A be a $N \times N$ matrix function such that

$$\begin{cases} A = (A_{ij})_{i,j\in\{1,\cdots,N\}} \in (L^\infty(\Omega))^{N\times N}, \\ \\ A(x)\xi\xi \geq \alpha|\xi|^2, \text{ a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \quad \text{for some } \alpha > 0. \end{cases}$$

We set

$$A' = (A_{ij})_{i,j=1,\dots,N-1}, \quad V' = (A_{iN})_{i=1,\cdots,N-1}, \quad H' = (A_{Nj})_{j=1,\cdots,N-1},$$

so the matrix A can be written as

$$A = \begin{pmatrix} A' & V' \\ H' & A_{NN} \end{pmatrix}.$$

For a.e. x in Ω, the system

$$A'(x)z'(x) + V'(x) = 0$$

admits the unique (column $(N-1)$-vector) solution

$$z'(x) = -(A'(x))^{-1}V'(x), \quad z' \in (L^\infty(\Omega))^{N-1}.$$

We define the coefficient a_0 by

$$a_0 = A_{NN} + H'z' = A_{NN} - H'(A')^{-1}V', \text{ a.e. in } \Omega.$$

Observe that

$$\begin{cases} a_0 \in L^\infty(\Omega), \\ \\ a_0(x) \geq \alpha, \text{ a.e. } x \in \Omega. \end{cases}$$

Remark *a_0 and z' depend only on the matrix A, and not on Ω and Ω_ε. In particular, $z' = 0$ and $a_0 = 1$, if $A = Id$.*

Anisotropic linear case

Let

$$c \in L^\infty(\Omega), \quad c(x) \geq \gamma, \text{ a.e. } x \in \Omega, \quad \text{for some } \gamma > 0,$$

and

$$f \in L^2(\Omega).$$

Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the unique solution to

$$\begin{cases} -\operatorname{div}(ADu_\varepsilon) + cu_\varepsilon = f, \text{ in } \Omega_\varepsilon, \\ \\ ADu_\varepsilon\, \nu_\varepsilon = 0, \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

where ν_ε denotes the unit outer normal on $\partial\Omega_\varepsilon$.

A.G., O. Guibé, and F. Murat (Arch. Rational Mech. Anal. - 2017) proved the following result.

Theorem

$$\|u^a_\varepsilon - u^a\|_{L^2(\Omega^a_\varepsilon)} \longrightarrow 0,$$

$$\|D_{x_N} u^a_\varepsilon - D_{x_N} u^a\|_{L^2(\Omega^a_\varepsilon)} \longrightarrow 0,$$

$$\|D_{x'} u^a_\varepsilon - D_{x_N} u^a \, z'\|_{(L^2(\Omega^a_\varepsilon))^{N-1}} \longrightarrow 0,$$

$$u^b_\varepsilon \longrightarrow u^b \textit{ strongly in } H^1(\Omega^b).$$

as ε tends to zero, where $u = (u^a, u^b) \in V^2(\Omega)$ is the unique weak solution to the following problem:

$$\begin{cases} -D_{x_N}\left(a_0 D_{x_N} u^a\right) + cu^a = f, \text{ in } \Omega^a, \\ \\ -\operatorname{div}(ADu^b) + cu^b = f, \text{ in } \Omega^b, \\ \\ u^a = u^b, \quad -\theta a_0 D_{x_N} u^a \nu_N^0 = ADu^b \nu^0, \text{ on } \Sigma^0, \\ \\ -a_0 D_{x_N} u^a \nu_N^a = 0, \text{ on } \Sigma^a, \\ \\ ADu^b \, \nu^b = 0, \text{ on } \partial\Omega^b \setminus \Sigma^0, \end{cases}$$

where ν^0 is the unit normal to Σ^0 outer to Ω^b (and inner to Ω^a), ν_N^0 its N-th component, ν_N^a is the N-th component of the unit normal to Σ^a outer to Ω^a, and ν^b the unit normal to $\partial\Omega^b \setminus \Sigma^0$ outer to Ω^b.

Corollary

$$\widetilde{u^a_\varepsilon} \rightharpoonup \theta u^a \text{ weakly in } L^2(\Omega^a),$$

$$\widetilde{D_{x_N} u^a_\varepsilon} = D_{x_N} \widetilde{u^a_\varepsilon} \rightharpoonup \theta D_{x_N} u^a \text{ weakly in } L^2(\Omega^a),$$

$$\widetilde{D_{x'} u^a_\varepsilon} \rightharpoonup \theta D_{x_N} u^a \, z' \text{ weakly in } (L^2(\Omega^a))^{N-1},$$

$$u^b_\varepsilon \longrightarrow u^b \text{ strongly in } H^1(\Omega^b).$$

as ε tends to zero.

The monotone case

Let $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ be the unique weak solution to the following problem

$$\begin{cases} -div(a(Du_\varepsilon)) + |u_\varepsilon|^{p-2}u_\varepsilon = f, \text{ in } \Omega_\varepsilon, \\ \\ a(Du_\varepsilon)\cdot\nu = 0, \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

where $p \in]1, +\infty[$, $f \in L^{\frac{p}{p-1}}(\Omega_\varepsilon)$ and $a = (a_1, \cdots, a_N) : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone continuous function such that

$$\begin{cases} \exists\alpha > 0,\ c_1 \geq 0 \ :\ \alpha|\xi|^p - c_1 \leq a(\xi)\xi, \quad \forall\xi \in \mathbb{R}^N, \\ \\ \exists\beta > 0, c_2 \geq 0 \ :\ |a(\xi)| \leq \beta|\xi|^{p-1} + c_2,\ \forall\xi \in \mathbb{R}^N. \end{cases}$$

D. Blanchard, L. Carbone and A.G. (M2AN, Math. Model. Numer. Anal. - 1999) proved the following result.

Theorem

As ε tends to zero,

$$\begin{cases} \widetilde{u}_\varepsilon \rightharpoonup |\omega| u^a \text{ weakly in } L^p(\Omega^a), \\ \widetilde{\dfrac{\partial u_\varepsilon}{\partial x_N}} = \dfrac{\partial \widetilde{u}_\varepsilon}{\partial x_N} \rightharpoonup |\omega| \dfrac{\partial u^a}{\partial x_N} \text{ weakly in } L^p(\Omega^a), \\ u_\varepsilon \rightharpoonup u^b \text{ weakly in } W^{1,p}(\Omega^b); \end{cases}$$

and, up to a subsequence,

$$\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} \rightharpoonup d_i \text{ weakly in } L^p(\Omega^a), \quad \forall i \in \{1, \cdots, N-1\},$$

where $\big((u^a, u^b), d_1, \cdots, d_{N-1}\big) \in V^p(\Omega) \times (L^p(\Omega^a))^{N-1}$, *with*

$$V^p(\Omega) =$$

$$\left\{ v = (v^a, v^b) \in L^p(\Omega^a) \times W^{1,p}(\Omega^b),\ \frac{\partial v^a}{\partial x_N} \in L^p(\Omega^a),\ v^a = v^b \text{ on } \Sigma^0 \right\}$$

solves the following system

$$
\begin{cases}
-\dfrac{\partial}{\partial x_N} a_N \left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{N-1}}{|\omega|}, \dfrac{\partial u^a}{\partial x_N} \right) + |u^a|^{p-2} u^a = f^a, \ \text{ in } \Omega^a, \\
-div(a(Du^b)) + |u^b|^{p-2} u^b = f^b, \ \text{in } \Omega^b, \\
u^a = u^b, \quad |\omega| a_N \left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{N-1}}{|\omega|}, \dfrac{\partial u^a}{\partial x_N} \right) = a_N(Du^b), \ \text{on } \Sigma^0, \\
a_N \left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{N-1}}{|\omega|}, \dfrac{\partial u^a}{\partial x_N} \right) = 0, \ \text{on } \Sigma^a, \\
a(Du^b) \cdot \nu = 0, \ \text{on } \partial\Omega^b \setminus \Sigma^0, \\
a_i \left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{N-1}}{|\omega|}, \dfrac{\partial u^a}{\partial x_N} \right) = 0, \ \text{in } \Omega^a, \quad \forall i \in 1, \cdots, N-1.
\end{cases}
$$

The couple (u^a, u^b) is unique. If a is strictly monotone, also $(d_1, \cdots, d_{N-1})$ is unique.

Moreover, the energies converge in the sense that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} a(Du_\varepsilon)Du_\varepsilon + |u_\varepsilon|^p\,dx$$

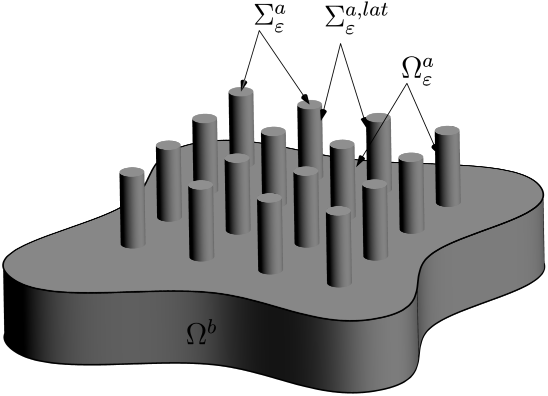
$$= |\omega|\int_{\Omega^a} a_N\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{N-1}}{|\omega|},\frac{\partial u^a}{\partial x_N}\right)\frac{\partial u^a}{\partial x_N} + |u^a|^p\,dx$$

$$+\int_{\Omega^b}\left(a(Du^b)Du^b + |u^b|^p\right)dx.$$

The limit problem for the p-Laplacian

$$
\begin{cases}
-\dfrac{\partial}{\partial x_N}\left(\left|\dfrac{\partial u^a}{\partial x_N}\right|^{p-2}\dfrac{\partial u^a}{\partial x_N}\right) + |u^a|^{p-2}u^a = f, \text{ in } \Omega^a, \\
-\mathrm{div}(|Du^b|^{p-2}Du^b) + |u^b|^{p-2}u^b = f, \text{ in } \Omega^b, \\
u^a = u^b, \;\; |\omega|\left|\dfrac{\partial u^a}{\partial x_N}\right|^{p-2} = |Du^b|^{p-2}\dfrac{\partial u^b}{\partial x_N}, \text{ on } \Sigma^0, \\
\dfrac{\partial u^a}{\partial x_N} = 0, \text{ on } \Sigma^a, \\
|Du^b|^{p-2}Du^b \cdot \nu = 0, \text{ on } \partial\Omega^b \setminus \Sigma^0, \\
d_1 = \cdots = d_{N-1} = 0, \text{ in } \Omega^a.
\end{cases}
$$

This result was already obtained by A. Corbo Esposito, P. Donato, A. G, and C. Picard in 1997.



Σ_ε^a
$\Sigma_\varepsilon^{a,lat}$
Ω_ε^a
Ω^b

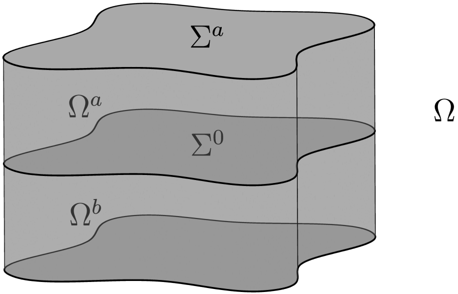


Figure: Ω

Signorini boundary value problem

$$
\begin{cases}
-\mathrm{div}(a(x, Du_\varepsilon(x))) + a_0(x, u_\varepsilon(x)) = f(x), \text{ in } \Omega_\varepsilon, \\[2ex]
u_\varepsilon = 0, \text{ on } \Sigma_\varepsilon^a, \\[2ex]
\left\{ \begin{array}{l} u_\varepsilon(x) \le g(x), \quad a(x, Du_\varepsilon(x))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon(x)) \le 0, \\[2ex] (u_\varepsilon(x) - g(x))(a(x, Du_\varepsilon(x))\nu_\varepsilon(x) + \varepsilon^\lambda h(x, u_\varepsilon)) = 0, \end{array} \right. \qquad \text{on } \Sigma_\varepsilon^{a,lat}, \\[2ex]
a(x, Du_\varepsilon(x))\nu_\varepsilon(x) = 0, \text{ on } \partial\Omega_\varepsilon \setminus \left(\Sigma_\varepsilon^a \cup \Sigma_\varepsilon^{a,lat} \right).
\end{cases}
$$

The Signorini boundary conditions in the third and fourth lines mean that on $\Sigma_\varepsilon^{a,lat}$ one can distinguish two *a priori* unknown subsets where u_ε satisfies the complementary boundary conditions:

$$u_\varepsilon(x) = g(x), \quad \text{or} \quad a(x, Du_\varepsilon(x))\nu_\varepsilon(x) = -\varepsilon^\lambda h(x, u_\varepsilon).$$

Such a problem can modelize chemical activity in a multi-structure with thick absorbers (for instance, the adsorption of nutrients on the tissues of the stomach wall). The impossibility to control physical processes on the boundary of the teeth suggests to use Signorini b. c. which seem more realistic for describing real phenomena.

Assumptions

$p \in [2, +\infty[$,

$$a : (x,\xi) \in \Omega \times \mathbb{R}^N \to a(x,\xi) = (a_1(x,\xi), \cdots, a_{N-1}(x,\xi), a_N(x,\xi))$$

$= (a'(x,\xi), a_N(x,\xi)) \in \mathbb{R}^N$ is a function such that

$$a \text{ is a Carathéodory function,}$$

$$a(x,\cdot) \text{ is strictly monotone for a.e. } x \in \Omega,$$

$$\exists \alpha \in]0,+\infty[,\ \alpha_1 \in L^1(\Omega) \ : \ \alpha|\xi|^p + \alpha_1(x) \le a(x,\xi)\xi, \text{ a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,$$

$$\exists \beta \in]0,+\infty[,\ \beta_1 \in L^{\frac{p}{p-1}}(\Omega) : |a(x,\xi)| \le \beta|\xi|^{p-1} + \beta_1(x), \text{ a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N$$

$$a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a function such that

$$a_0 \text{ is a Carathéodory function,}$$

$$a_0(x,\cdot) \text{ is monotone for a.e. } x \in \Omega,$$

$$\exists \gamma \in]0,+\infty[,\ \gamma_1 \in L^{\frac{p}{p-1}}(\Omega) : |a_0(x,t)| \le \gamma|t|^{p-1} + \gamma_1(x), \text{ a.e. } x \in \Omega,\ \forall t \in \mathbb{R};$$

$$h : \Omega^a \times \mathbb{R} \to \mathbb{R}$$

is a function such that

is a continuous function,

$$h(x, \cdot) \text{ is monotone for a.e. } x \in \Omega^a,$$

$$\exists \eta \in]0, +\infty[,\ \eta_1 \in W^{1,\frac{p}{p-1}}(\Omega^a)\ :\ |h(x,t)| \leq \eta |t|^{p-1} + \eta_1(x), \quad \text{a.e. } x \in \Omega^a,$$

$$\exists Dh \text{ and it is a Carathéodory valued-function,}$$

$$\begin{cases} \exists \theta \in]0, +\infty[,\ \theta_1 \in L^{\frac{p}{p-1}}(\Omega^a)\ :\ |D_t h(x,t)| \leq \theta |t|^{p-2}, \\ \\ |D_{x_i} h(x,t)| \leq \theta |t|^{p-1} + \theta_1(x), \quad \text{a.e. } x \in \Omega^a, \quad \forall t \in \mathbb{R}, \quad \forall i \in \{1, \cdots, N\}; \end{cases}$$

$$f \in L^{\frac{p}{p-1}}(\Omega),$$

$$g \in W^{1,p}(\Omega^a), \quad g \geq 0 \text{ a.e. in } \Omega^a, \quad g_{|_{\Sigma^a \cup \Sigma^0}} = 0,$$

and

$$\lambda \in \mathbb{R}.$$

Some examples

- h is a constant

- $h(t,x) = g_1(x)|t|^{p-2}t + g_2(x)$

with $p \geq 2, \quad g_1 \in W^{1,\infty}(\Omega^a),$ and $g_2 \in W^{1,\frac{p}{p-1}}(\Omega^a)$

- In Langmuir adsorption model $p = 2$ and

$$h(t) = \frac{t}{1+|t|}.$$

The weak formulation of such a problemis given by the following variational inequality

$$\begin{cases} u_\varepsilon \in \mathcal{K}_\varepsilon = \left\{ v \in W^{1,p}(\Omega_\varepsilon) : v \le g \text{ on } \Sigma_\varepsilon^{a,lat}, \quad v_{|_{\Sigma_\varepsilon^{a}}} = 0 \right\}, \\ \displaystyle\int_{\Omega_\varepsilon} a(x, Du_\varepsilon) D(v - u_\varepsilon) dx + \int_{\Omega_\varepsilon} a_0(x, u_\varepsilon)(v - u_\varepsilon) dx \\ \displaystyle +\varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} h(x, u_\varepsilon)(v - u_\varepsilon) d\sigma \ge \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx, \quad \forall v \in \mathcal{K}_\varepsilon. \end{cases}$$

Let

$$\mathcal{K} = \Big\{ \quad v = (v^a, v^b) \in L^p(\Omega^a) \times W^{1,p}(\Omega^b) \ : \ D_{x_N} v^a \in L^p(\Omega^a),$$
$$v^a \leq g \text{ a.e. in } \Omega^a, \quad v^a_{|_{\Sigma^a}} = 0, \quad v^a_{|_{\Sigma^0}} = v^b_{|_{\Sigma^0}} \Big\}.$$

A.G. and T. A. Mel'nyk (J. Differential Equations - 2018) proved the following result.

Theorem

Assume $\lambda \geq 1$, Assume ω having C^{m+2}-regularity with $m > \frac{N-1}{2}$. For every ε, let u_ε be the unique solution to our problem and set $u_\varepsilon^a = u_{\varepsilon}|_{\Omega_\varepsilon^a}$, $u_\varepsilon^b = u_{\varepsilon}|_{\Omega^b}$. Then,

$$\widetilde{u_\varepsilon^a} \rightharpoonup |\omega| u^a \text{ weakly in } L^p(\Omega^a),$$

$$\widetilde{D_{x_N} u_\varepsilon^a} = D_{x_N} \widetilde{u_\varepsilon^a} \rightharpoonup |\omega| D_{x_N} u^a \text{ weakly in } L^p(\Omega^a),$$

$$\widetilde{D_{x'} u_\varepsilon^a} \rightharpoonup |\omega| d' \text{ weakly in } (L^p(\Omega^a))^{N-1},$$

$$u_\varepsilon^b \rightharpoonup u^b \text{ weakly in } W^{1,p}(\Omega^b),$$

$$a\widetilde{(x, Du_\varepsilon^a)} \rightharpoonup |\omega| \left(0, \cdots, 0, a_N\left(x, (d', D_{x_N} u^a)\right)\right) \text{ weakly in } (L^{\frac{p}{p-1}}(\Omega^a))^N,$$

$$a(x, Du_\varepsilon^b) \rightharpoonup a(x, Du^b) \text{ weakly in } (L^{\frac{p}{p-1}}(\Omega^b))^N,$$

$$a_0\widetilde{(x, u_\varepsilon^a)} \rightharpoonup |\omega| a_0(x, u^a) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a),$$

$$a_0(x, u_\varepsilon^b) \rightharpoonup a_0(x, u^b) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^b),$$

$$h\widetilde{(x, u_\varepsilon^a)} \rightharpoonup |\omega| h(x, u^a) \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a),$$

as ε tends to zero, and (u^a, u^b) and d' is the unique solution *to the following system*:

$$
\begin{cases}
(u^a, u^b) \in \mathcal{K}, \quad d' \in (L^p(\Omega^a))^{N-1}, \\
a'\left(x, (d', D_{x_N} u^a)\right) = 0, \text{ a.e. in } \Omega^a, \\
|\omega| \displaystyle\int_{\Omega^a} \left(a_N\left(x, (d', D_{x_N} u^a)\right)\left(D_{x_N} v - D_{x_N} u^a\right) + a_0(x, u^a)(v - u^a)\right) dx \\
+\delta_{\lambda,1} |\partial\omega| \displaystyle\int_{\Omega^a} h(x, u^a)(v - u^a) dx \\
+ \displaystyle\int_{\Omega^b} \Big(a(x, Du^b)(Dv - Du^b) + a_0(x, u^b)(v - u^b) \Big) \, dx \\
\geq |\omega| \displaystyle\int_{\Omega^a} f(v - u^a) dx + \int_{\Omega^b} f(v - u^b) dx, \quad \forall v \in \mathcal{K},
\end{cases}
$$

where $\delta_{\lambda,1}$ is the Kronecker delta.

Remarks

- The limit problem admits a unique solution $(u^a, u^b) \in \mathcal{K}$, $d' \in (L^p(\Omega^a))^{N-1}$. This problem is composed by the algebraic system

$$a'\left(x, (d'(x), D_{x_N} u^a(x))\right) = 0, \text{ a.e. in } \Omega^a,$$

with $N - 1$ equations and N unknowns (d', u^a), coupled to a variational inequality involving (d', u^a, u^b), with u^a and u^b satisfying a transmission condition on Σ^0.

- If $\lambda > 1$, the Signorini boundary condition does not give any contribution to the limit problem.

- If $\lambda = 1$, the Signorini boundary condition becomes the volume integral

$$|\partial\omega| \int_{\Omega^a} h(x, u^a)(v - u^a)dx,$$

in the limit problem.

An auxiliary problem

Let Ξ be the unique weak solution to the following problem

$$\begin{cases} \Delta \Xi = \dfrac{|\partial \omega|}{|\omega|}, \text{ in } \omega, \\ D\Xi \cdot \nu = 1, \text{ on } \partial\omega, \\ \displaystyle\int_\omega \Xi dy = 0, \end{cases}$$

where ν denotes the unit outer normal on $\partial\omega$. Note that Ξ belongs to $C^2(\overline{\omega})$, since ω has C^{m+2}-regularity with $m > \frac{N-1}{2}$. Consequently Ξ is also a classical solution. In what follows, we set

$$C_\Xi = \sup_{\overline{\omega}} |D\Xi|.$$

Lemma

Let Ξ be denoting also the $]0,1[^{N-1}$- periodic extension to $\cup_{k\in\mathbb{Z}^{N-1}}(\overline{\omega}+k)$ of the solution of previous problem. Then,

$$\varepsilon \int_{\Sigma_\varepsilon^{a,lat}} v d\sigma = \frac{|\partial\omega|}{|\omega|}\int_{\Omega_\varepsilon^a} v dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'} v dx, \quad \forall v \in W^{1,1}(\Omega_\varepsilon^a).$$

Example of using previous lemma: $u^a \leq g$, a.e. in Ω^a.

Proof.

Since $u_\varepsilon \in \mathcal{K}_\varepsilon$, one has

$$\varepsilon \int_{\Sigma_\varepsilon^{a,lat}} u_\varepsilon^a \varphi d\sigma \leq \varepsilon \int_{\Sigma_\varepsilon^{a,lat}} g\varphi d\sigma, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0 \text{ in } \Omega^a, \quad \forall \varepsilon.$$

Thanks to previous Lemma, this inequality is equivalent to

$$\frac{|\partial\omega|}{|\omega|} \int_{\Omega^a} \widetilde{u_\varepsilon^a} \varphi dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'}(u_\varepsilon^a \varphi) dx$$

$$\leq \frac{|\partial\omega|}{|\omega|} \int_{\Omega^a} \chi_{\Omega_\varepsilon^a} g\varphi dx + \varepsilon \int_{\Omega_\varepsilon^a} (D\Xi)\left(\frac{x'}{\varepsilon}\right) D_{x'}(g\varphi) dx, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0$$

Passing to the limit, as ε tends to zero, in this inequality gives

$$\int_{\Omega^a} u^a \varphi dx \leq \int_{\Omega^a} g\varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega^a) : \varphi \geq 0 \text{ in } \Omega^a,$$

which completes the proof. □

The case $\lambda < 1$

A.G. and T. A. Mel'nyk, work in progress

If $\lambda < 1$, the existence and the uniqueness of the solution to our ε-problem holds agains true, but one can not expect to obtain *a priori* estimates independent of ε, without additional assumptions on h, as the following example shows.

Let u_ε be the unique weak solution to

$$\begin{cases} -\Delta u_\varepsilon = 0, \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0, \text{ on } \Sigma_\varepsilon^a, \\ \begin{cases} u_\varepsilon(x) \leq g(x), \quad Du_\varepsilon(x)\cdot\nu_\varepsilon(x) + \varepsilon^\lambda \leq 0, \\ (u_\varepsilon(x) - g(x))(Du_\varepsilon(x)\cdot\nu_\varepsilon(x) + \varepsilon^\lambda) = 0, \end{cases} \quad \text{on } \Sigma_\varepsilon^{a,lat}, \\ Du_\varepsilon(x)\cdot\nu_\varepsilon(x) = 0, \text{ on } \partial\Omega_\varepsilon \setminus \left(\Sigma_\varepsilon^a \cup \Sigma_\varepsilon^{a,lat}\right), \end{cases}$$

where $\lambda \in [-\infty, 1[$ and $g \in H^1(\Omega^a)$ is a non-negative function with $g_{|_{\Sigma^a \cup \Sigma^0}} = 0$, i.e.

$$\begin{cases} u_\varepsilon \in \mathcal{K}_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon, \Sigma_\varepsilon^a) : v \le g \text{ on } \Sigma_\varepsilon^{a,lat} \right\}, \\ \displaystyle\int_{\Omega_\varepsilon} Du_\varepsilon \cdot D(v - u_\varepsilon) dx + \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} (v - u_\varepsilon) d\sigma \ge 0, \quad \forall v \in \mathcal{K}_\varepsilon. \end{cases} \tag{1}$$

Then,

$$\exists \varepsilon_0 \in]0,1[, \ \exists \mu_1, \ \mu_2 \in]0, +\infty[\ : \ \frac{\mu_1}{\varepsilon^{1-\lambda}} \le \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le \frac{\mu_2}{\varepsilon^{1-\lambda}}, \quad \forall \varepsilon \in]0, \varepsilon_0[. \tag{2}$$

Let us prove the first inequality in (2). Let $\psi \in C_0^\infty(\Omega^a)$ be a non-positive and not identically zero function. Then, choosing $v = \psi + u_\varepsilon$ as test function in (1) gives

$$\int_{\Omega_\varepsilon^a} Du_\varepsilon \cdot D\psi dx + \varepsilon^\lambda \int_{\Sigma_\varepsilon^{a,lat}} \psi d\sigma \ge 0, \quad \forall \varepsilon.$$

Consequently, thanks to the Cauchy-Schwarz inequality and previous Lemma, one obtains

$$\|D\psi\|_{L^2(\Omega^a)} \|u_\varepsilon\|_{H^1(\Omega)} \ge \varepsilon^{\lambda-1} \frac{|\partial\omega'|}{|\omega'|} \int_{\Omega_\varepsilon^a} (-\psi) dx - \varepsilon^\lambda \int_{\Omega_\varepsilon^a} (D\Xi) \left(\frac{x'}{\varepsilon} \right) \cdot D_{x'} \psi dx,$$

which implies the first inequality in (2) since

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\lambda-1}\dfrac{|\partial\omega'|}{|\omega'|}\displaystyle\int_{\Omega^a_\varepsilon}(-\psi)dx-\varepsilon^{\lambda}\int_{\Omega^a_\varepsilon}(D\Xi)\left(\frac{x'}{\varepsilon}\right)\cdot D_{x'}\psi dx}{\varepsilon^{\lambda-1}}=$$

$$=|\partial\omega'|\int_{\Omega^a}(-\psi)dx>0,$$

and $\|D\psi\|_{L^2(\Omega^a)}\neq 0$ (note that ψ can not be a constant function).

In the case $\lambda < 1$, we assume the following further hypothesis which guarantees us *a priori* estimates of u_ε, independently of ε:

$$\exists \rho \in]0, +\infty[: \ \rho |t|^p \leq h(x,t)t, \text{ a.e. } x \in \Omega^a, \ \ \forall t \in \mathbb{R}.$$

Theorem

Let $\lambda \in]0,1[$. Then, under all previous assumptions, as $\varepsilon \to 0$,

$$\widetilde{u_\varepsilon^a} \to 0 \text{ strongly in } L^p(\Omega^a), \quad \partial_{x_N} \widetilde{u_\varepsilon^a} \rightharpoonup 0, \text{ weakly in } L^p(\Omega^a), \tag{3}$$

$$\widetilde{D_{x'} u_\varepsilon^a} \rightharpoonup |\omega'| d' \text{ weakly in } (L^p(\Omega^a))^{N-1},$$
$$u_\varepsilon^b \rightharpoonup u^b \text{ weakly in } W^{1,p}(\Omega^b),$$

where d' is the unique solution to the following algebraic system

$$a' \left(x, (d'(x), 0)\right) = 0, \text{ a.e. in } \Omega^a,$$

and u^b is the unique weak solution to the following problem

$$\begin{cases} -\mathrm{div}(a(x, Du^b(x))) + a_0(x, u^b(x)) = f(x), \text{ in } \Omega^b, \\ \\ u^b = 0, \text{ on } \Sigma^0, \\ a(x, Du^b(x)) \cdot \nu(x) = 0, \text{ on } \partial\Omega^b \setminus \Sigma^0. \end{cases} \tag{4}$$

When $\lambda \leq 0$, convergences in (3) continue to hold true, but we are not able to identify the other limits. To overcome this gap, we add also an additonal assumption on a, precisely we assume

$$\exists \alpha \in]0, +\infty[: \quad \alpha |\xi|^p \leq a(x, \xi)\xi, \quad \text{a.e. } x \in \Omega^a, \quad \forall \xi \in \mathbb{R}^N. \qquad (5)$$

Theorem

Let $\lambda \leq 0$. Then, under all previous assumptions,

$$\widetilde{u_\varepsilon^a} \to 0 \textit{ strongly in } L^p(\Omega^a),$$

$$\widetilde{Du_\varepsilon^a} \rightharpoonup 0, \textit{ weakly in } \left(L^p(\Omega^a)\right)^N,$$
$$u_\varepsilon^b \rightharpoonup u^b \textit{ weakly in } W^{1,p}(\Omega^b),$$

as ε tends to zero, and u^b is the unique weak solution to problem (4).

This theorem holds true for any $\lambda < 1$, but when $\lambda \in]0, 1[$ this theorem is just a corollary of the previous one since the additional assumption (5) implies $a(x, 0) = 0$, consequently $d' = 0$, since the algebraic system $a' \left(x, (d'(x), 0)\right) = 0$ admits a unique solution due to the strictly monotonicity of a in the second variable.

Remarks

We explicitly note that the limit problem obtained for $\lambda < 1$ is the same obtained by homogenizing the following problem

$$\begin{cases} -\mathrm{div}(a(x, Du_\varepsilon(x))) + a_0(x, u_\varepsilon(x)) = f(x), \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0, \text{ on } \Sigma_\varepsilon^a \cup \Sigma_\varepsilon^{a,lat}, \\ a(x, Du_\varepsilon(x)) \cdot \nu_\varepsilon(x) = 0, \text{ on } \partial\Omega_\varepsilon \setminus \left(\Sigma_\varepsilon^a \cup \Sigma_\varepsilon^{a,lat}\right). \end{cases}$$

Roughly speaking,

• $\lambda = 1$ is a critical size and the new term

$$|\partial\omega| \int_{\Omega^a} h(x, u^a)(v - u^a)dx,$$

appears in the limit equation;

• the case $\lambda < 1$ is equivalent to homogenizing a homogeneous Dirichlet boundary value problem on $\Sigma_\varepsilon^a \cup \Sigma_\varepsilon^{a,lat}$;

• the case $\lambda > 1$ is equivalent to homogenizing a Signorini boundary value problem with homogeneous Neumann boundary constraint on $\Sigma_\varepsilon^{a,lat}$.

More general geometries of the brush

A.G., O. Guibé, and F. Murat (Arch. Rational Mech. Anal. - 2017)

Let $N \in \mathbb{N}$, $N \geq 2$,

$$\psi^a,\ \psi^0,\ \psi^b : \mathbb{R}^{N-1} \to \mathbb{R}$$

be three functions, with ψ^a and ψ^b continuous and ψ^0 Lipschitz continuous, such that

$$\psi^b(x') < \psi^0(x') < \psi^a(x'), \quad \forall x' \in \mathbb{R}^{N-1},$$

$\Omega' \subset \mathbb{R}^{N-1}$ be an open bounded set (no regularity is assumed on $\partial\Omega'$), and set

$$\begin{cases} \Omega^a = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x' \in \Omega',\ \psi^0(x') < x_N < \psi^a(x')\right\}, \\ \\ \Sigma^0 = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x' \in \Omega',\ x_N = \psi^0(x')\right\}, \\ \\ \Omega^b = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x' \in \Omega',\ \psi^b(x') < x_N < \psi^0(x')\right\}, \\ \\ \Omega = \Omega^a \cup \Sigma^0 \cup \Omega^b. \end{cases}$$

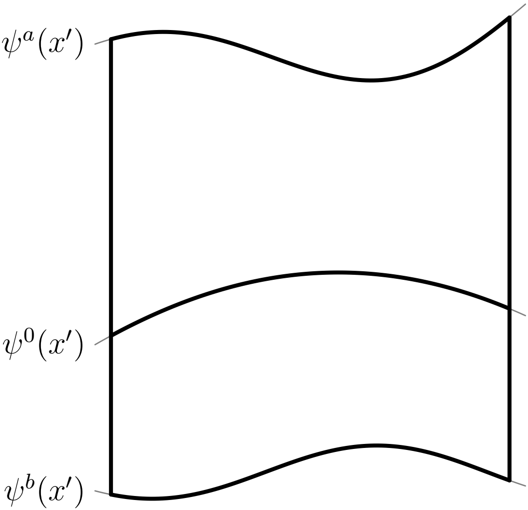


Figure: Ω in 2D

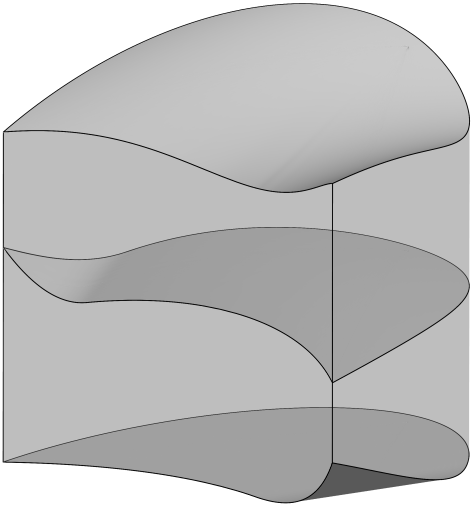


Figure: Ω in 3D

For every ε, let

$$\omega'_\varepsilon = \bigcup_{j \in J_\varepsilon} \omega^j_\varepsilon,$$

where $J_\varepsilon \subset \mathbb{N}$ is a (finite or infinite) family of indices, and $\omega^j_\varepsilon \subset \mathbb{R}^{N-1}$ is an open bounded set such that

$$\begin{cases} \operatorname{diam}(\omega^j_\varepsilon) \leq \varepsilon & \forall j \in J_\varepsilon, \\ \omega^j_\varepsilon \cap \omega^k_\varepsilon = \emptyset & \forall j, k \in J_\varepsilon, \quad j \neq k. \end{cases}$$

Moreover, we assume that

$$\chi_{\omega'_\varepsilon} \rightharpoonup \theta \text{ weakly-star in } L^\infty(\mathbb{R}^{N-1}), \text{ as } \varepsilon \to 0,$$

with

$$\theta(x') \geq \theta_0 > 0, \text{ a.e. } x' \in \mathbb{R}^{N-1}.$$

No periodicity assumption is required and no regularity on $\partial\omega^j_\varepsilon$ needs.

For every ε, let Q_ε be the "forest of cylinders" with cross-sections ω^j_ε, $j \in J_\varepsilon$, and infinite height, i.e.

$$Q_\varepsilon = \omega'_\varepsilon \times \mathbb{R}.$$

We put

$$\Omega^a_\varepsilon = Q_\varepsilon \cap \Omega^a, \quad \Sigma^0_\varepsilon = Q_\varepsilon \cap \Sigma^0, \quad \Omega_\varepsilon = \Omega^a_\varepsilon \cup \Sigma^0_\varepsilon \cup \Omega^b.$$

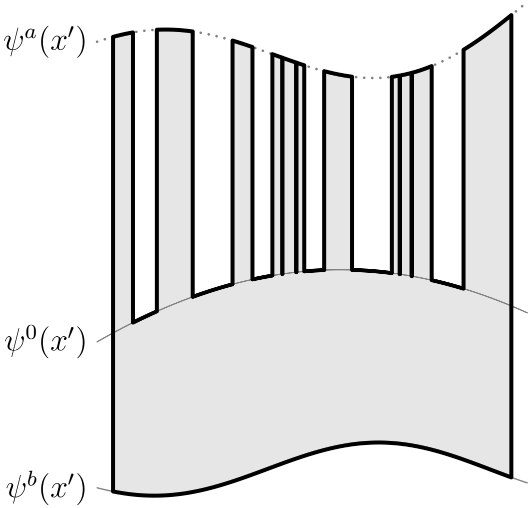


Figure: The comb in 2D

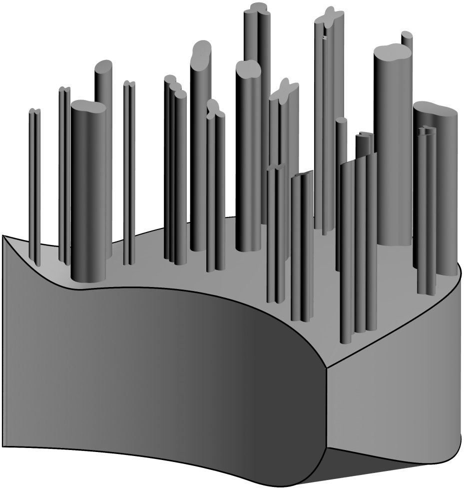


Figure: The brush in 3D

Remark *Previous assumptions on the cross-sections of the cylinders are satisfied in the periodic case. Namely, let*

$$l_i > 0, \quad i = 1, \cdots, N-1, \quad Y' = \prod_{i=1}^{N-1}]0, l_i[,$$

and let $\omega' \subseteq Y'$ *be an open set. Then* ω_ε^k *defined as*

$$\omega_\varepsilon^k = \varepsilon\omega' + \varepsilon(k_1 l_1, \cdots, k_{N-1} l_{N-1}), \quad \forall k = (k_1, \cdots, k_{N-1}) \in \mathbb{Z}^{N-1},$$

and ω'_ε *defined as*

$$\omega'_\varepsilon = \bigcup_{k \in \mathbb{Z}^{N-1}} \omega_\varepsilon^k$$

satisfy previous assumptions with

$$\theta = \frac{|\omega'|}{|Y'|}.$$

One can have $\omega' = Y'$*; then* $\theta = 1$*.*

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