On similarity of multi-scale procedures for thin and periodic structures

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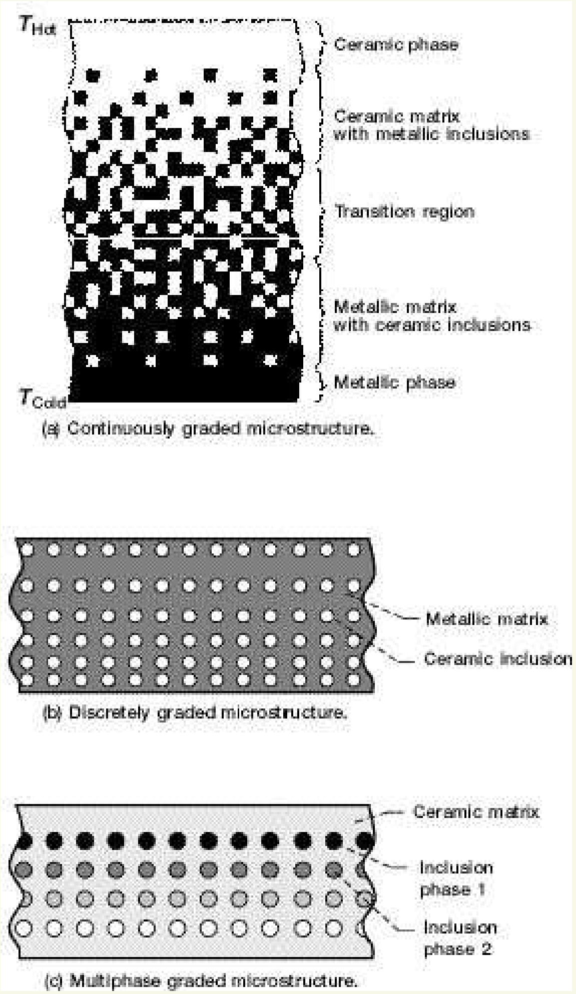
Functionally graded microstructures

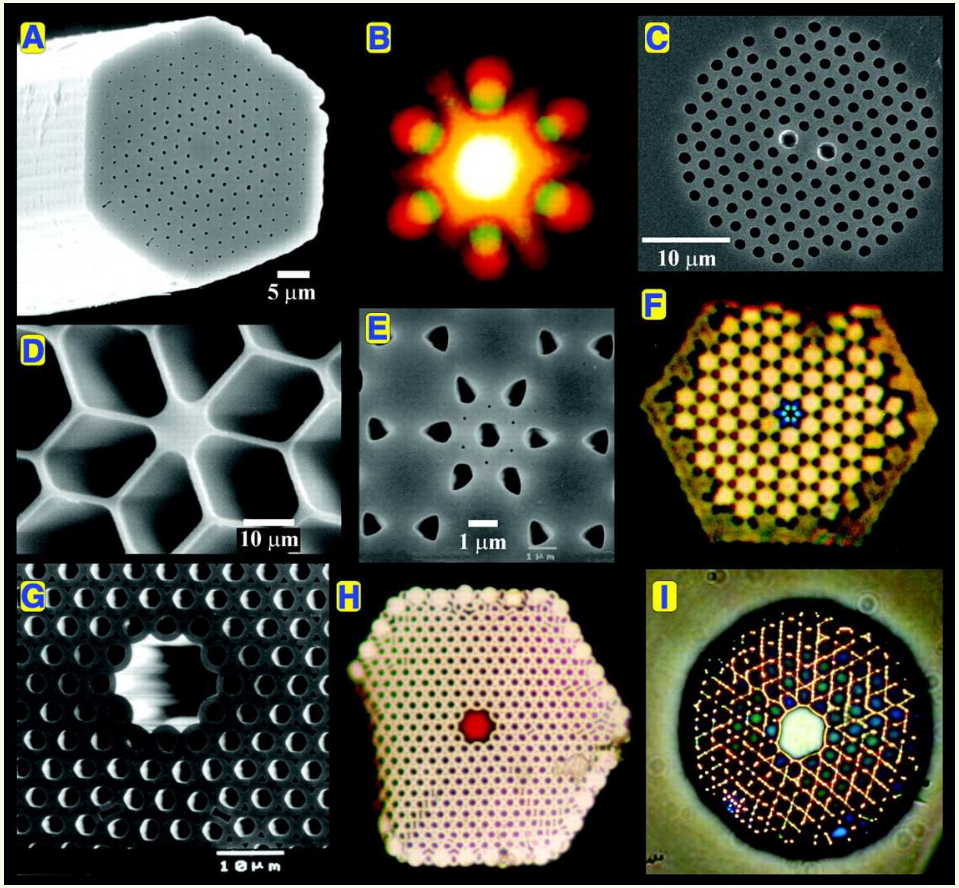


the picture is taken from NASA webpage

Civil structures

the picture is taken from http://www.wikipedia.org

Photonic crystals

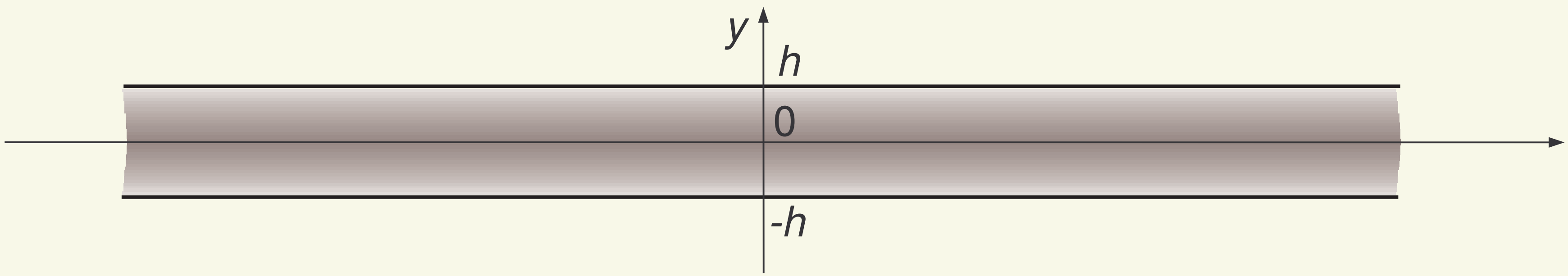


the picture is taken from the review of P Russell, Science 2003

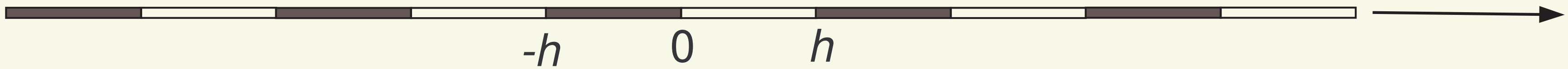
Two toy problems

The goal is to demonstrate the similarities of homogenization procedures for 2D thin functionally graded structures and 1D periodic structures, see R.V. Craster, L.M. Joseph & J. Kaplunov in Wave Motion 2014

(A) SH waves in a functionally graded layer (2D problem)



(B) Longitudinal waves in a periodic rod (1D problem)



Dynamic homogenization

Problem A

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\omega^2}{c^2(y)} u = 0$$

where $u = u(x, y)$

traction free faces

$\partial u / \partial y|_{y=\pm h} = 0$

Problem B

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\omega^2}{c^2(x)} u = 0$$

where $u = u(x)$

periodicity

$c(x) = c(x + 2h)$

Small parameter

$\epsilon = h/L \ll 1$ (L is typical wavelength along x-axis)

Scaling

$X = x/L, \qquad \xi = \alpha/h,$ where

$\alpha = y$ (Problem A)

$\alpha = x$ (Problem B)

Dimensionless equations in $u(X,\xi)$

$$u_{\xi\xi} + \epsilon^2 u_{XX} + \frac{\lambda^2}{C^2(\xi)} u = 0$$

$$u_{\xi\xi} + \underbrace{2\epsilon u_{X\xi}}_{\text{the only difference}} + \epsilon^2 u_{XX} + \frac{\lambda^2}{C^2(\xi)} u = 0$$

with $\lambda = \frac{\omega h}{c_0}$ and $C(\xi) = \frac{c(\xi)}{c_0}$

Classical low frequency limit ($\lambda \sim \epsilon$)

$$u(X,\xi) = u_0(X,\xi) + \epsilon u_1(X,\xi) + \epsilon^2 u_2(X,\xi) + \dots$$

and

$$\lambda^2 = \epsilon^2 \left(\lambda_0^2 + \epsilon \lambda_1^2 + \epsilon^2 \lambda_2^2 + \dots\right)$$

with Neumann boundary conditions

$$u_{i\xi}\big|_{\xi=\pm 1} = 0$$

with periodicity conditions

$$u_i(X,1) = u_i(X,-1),$$
$$u_{i\xi}(X,1) = u_{i\xi}(X,-1)$$

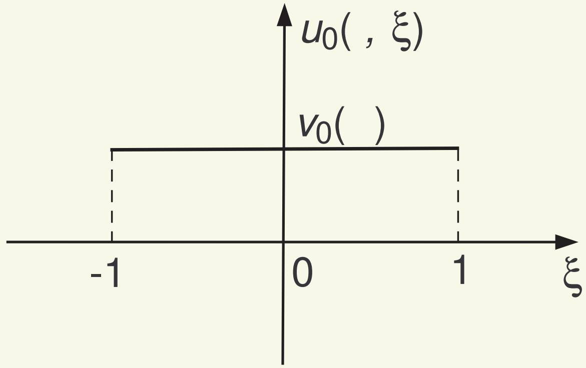
Dynamic homogenization

At leading order we get over a microscale

$$u_{0\xi\xi} = 0$$

resulting in uniform static variation along thickness or cell

$$u_0(X, \xi) = v_0(X).$$



Proceeding to higher orders

$$u_1(X, \xi) = 0, \quad \lambda_1 = 0 \text{ and } u_{2\xi\xi} = -v_{0XX} - \frac{\lambda_2^2}{C^2(\xi)} v_0$$

Finally, we arrive at 1D homogenized equation

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}x^2} + \frac{\omega^2}{\langle c \rangle^2} v_0 = 0, \text{ with } \langle c \rangle = \left[\frac{1}{2h} \int_{-h}^{h} c^{-2}(z) \mathrm{d}z \right]^{-1/2}$$

Non-classical high frequency limit $(\lambda \sim 1)$

The so-called high frequency long wave theory for thin elastic structures established some time ago (*see e.g. J.D.Kaplunov, L.Yu.Kossovich & E.V.Nolde, Dynamics of Thin Walled Elastic Bodies, Academic Press, N.-Y. 1998*) inspired a more recently developed high frequency homogenization procedure (*see R.V.Craster, J.Kaplunov & A.V.Pichugin, Proc R Soc A 2010)*

At leading order $u_0(X,\xi) = v_0(X)U_0(\xi)$ and

$$U_{0\xi\xi} + \frac{\lambda_0^2}{C^2(\xi)} U_0 = 0$$

with Neumann boundary conditions

$U_{0\xi}|_{\xi=\pm 1} = 0$

with periodicity conditions

$U_0(X,1) = U_0(X,-1)$,
$U_{0\xi}(X,1) = U_{0\xi}(X,-1)$

or antiperiodicity conditions (leading to periodicity with a **double** period)

$U_0(X,1) = -U_0(X,-1)$,
$U_{0\xi}(X,1) = -U_{0\xi}(X,-1)$

Dynamic homogenization

To certain extent there is an analogy between

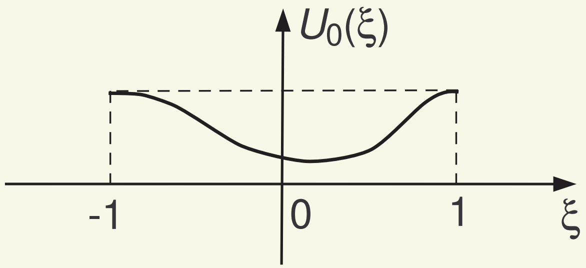
symmetry (antisymmetry) in ξ
for even $C^2(\xi)$

periodicity (antiperiodicity) in ξ

Eigenvalues λ_0 correspond to

thickness resonances

cell resonances



The sought for 1D homogenized equation is

$$h^2 T v_{0xx} + (\lambda^2 - \lambda_0^2) v_0 = 0$$

$$T = \frac{\int_{-h}^{h} U_0^2(z) C^{-2}(z) \mathrm{d}z}{\int_{-h}^{h} U_0^2(z) \mathrm{d}z}$$

T takes slightly more complicated form

(see R.V.Craster, J.Kaplunov & A.V.Pichugin, Proc R Soc A 2010)

Floquet-Bloch waves

$$\begin{bmatrix} u(-1) \\ u_\xi(-1) \end{bmatrix} = \exp(i2\kappa\varepsilon) \begin{bmatrix} u(1) \\ u_\xi(1) \end{bmatrix}$$

where κ - Bloch parameter.

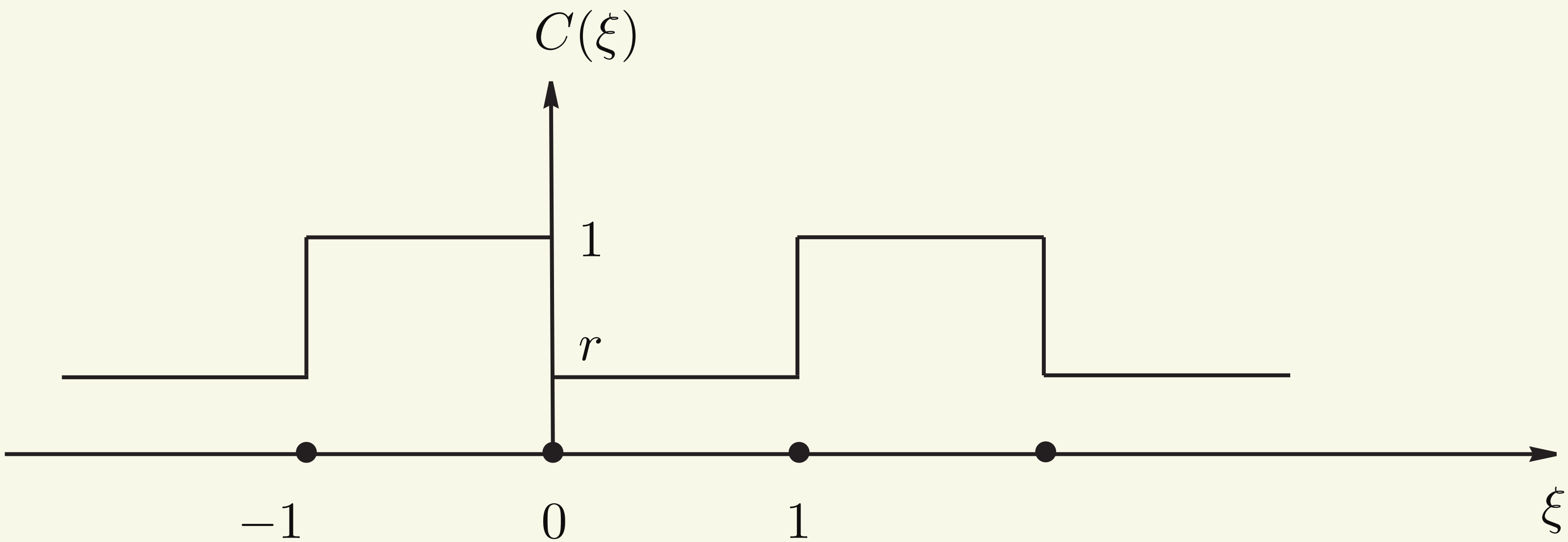
Bloch spectra $\lambda(\kappa)$ near edges of stop bands

$\kappa = 0$ — almost periodic solutions

$\kappa = \dfrac{\pi}{2\varepsilon}$ — almost anti-periodic solutions

Dynamic homogenization

a) piecewise uniform sound speed (constant coefficients)

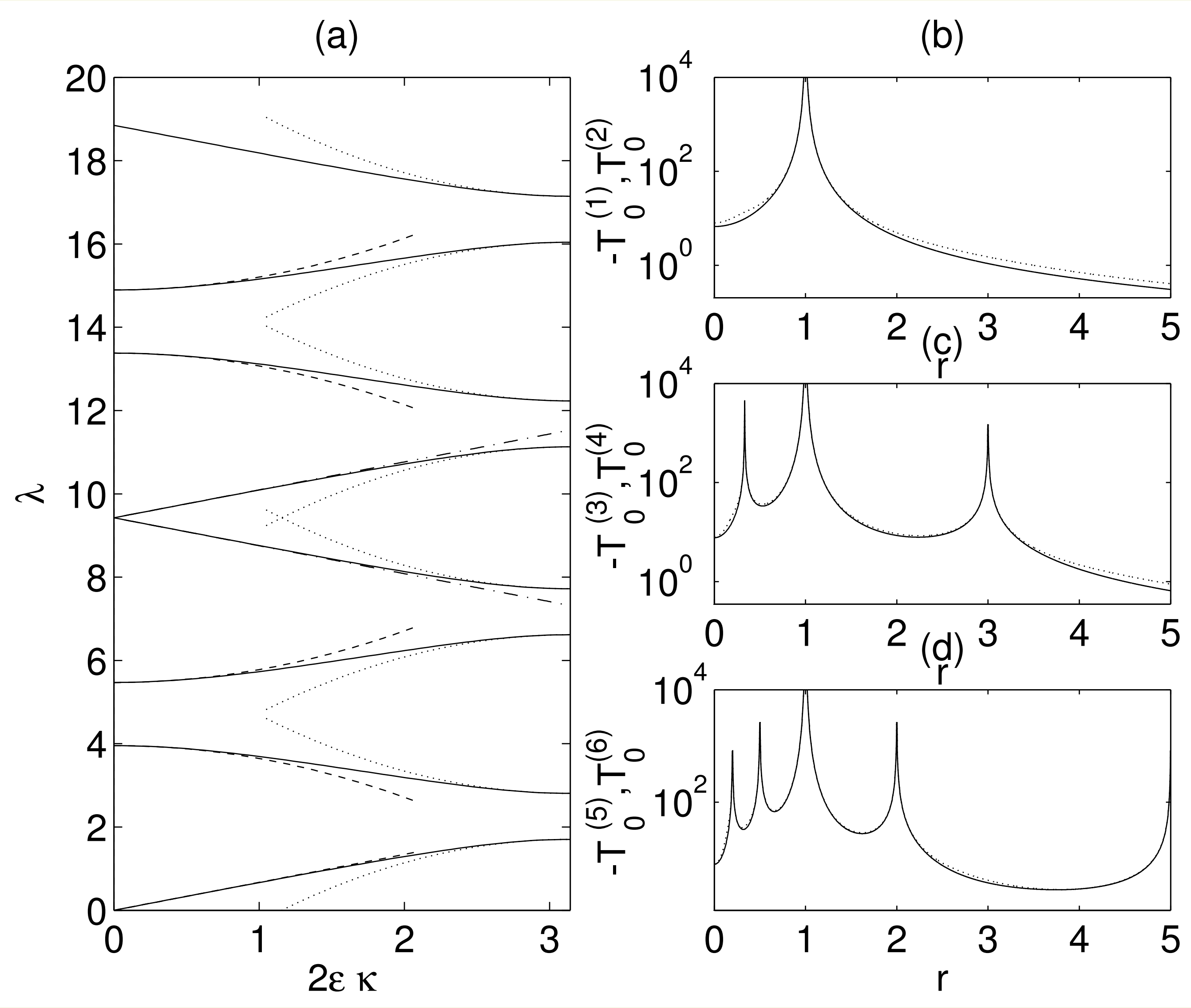


b) Mathieu's equation (variable coefficients)

$$C^{-2}(\xi) = \alpha - 2\theta \cos \xi$$

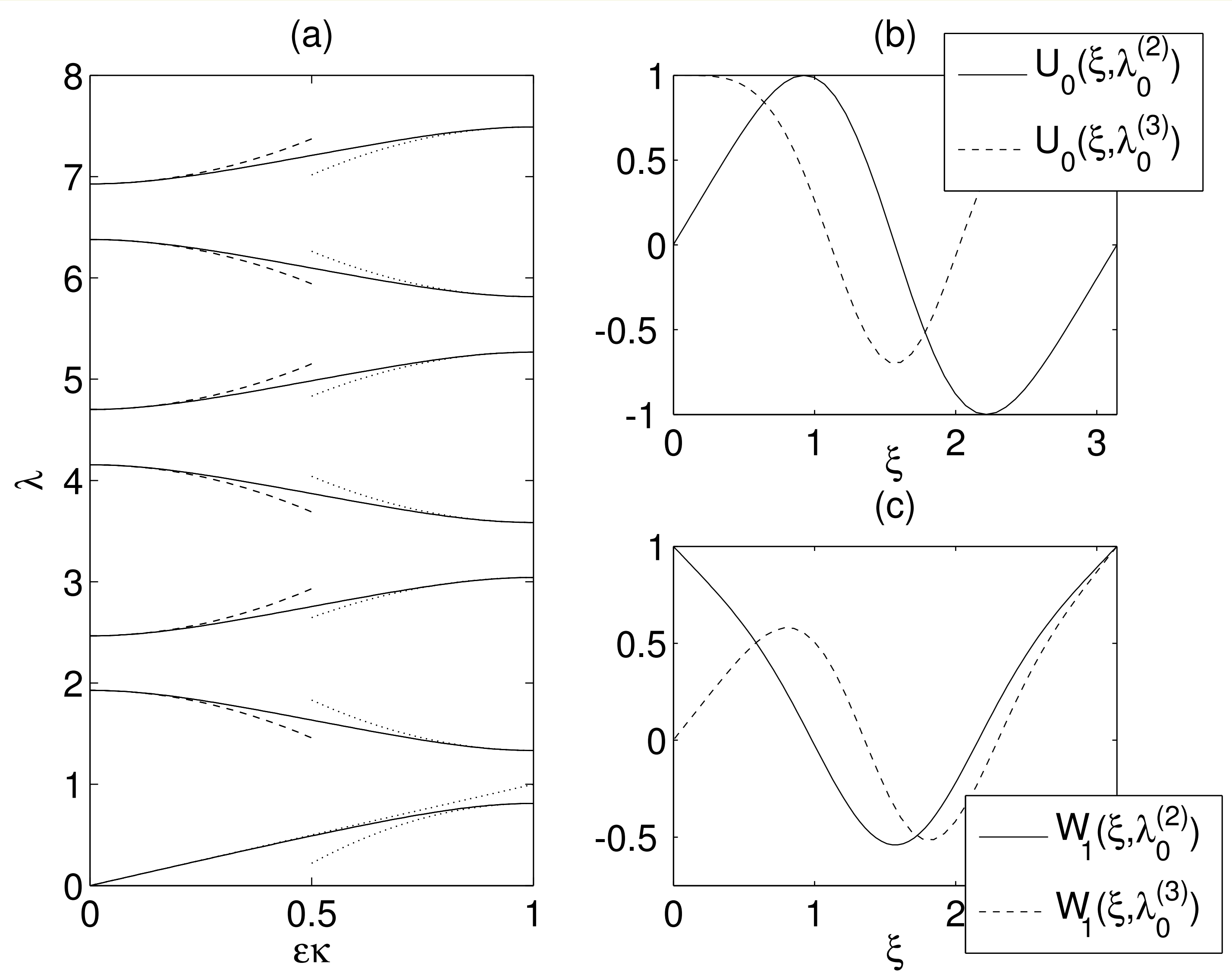
Dynamic homogenization

Piecewise uniform string $(r = 1/3)$



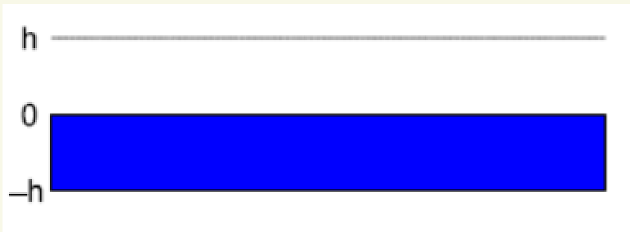
Dynamic homogenization

Mathieu's equation $(\alpha = 1, \theta = 1/2)$

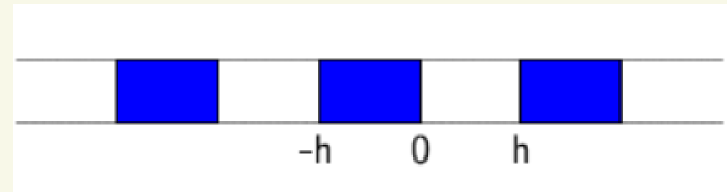


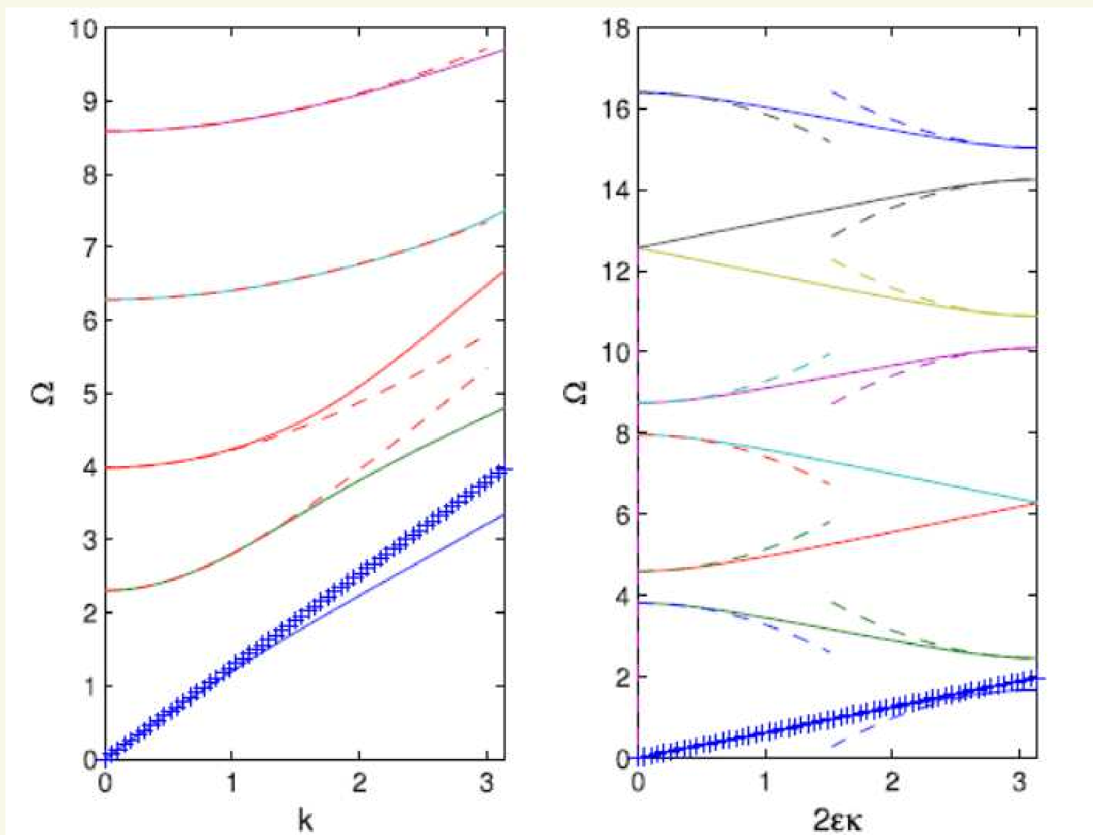
Dynamic homogenization

Rayleigh-Lamb vs Floquet-Bloch $(r = \frac{1}{2})$



vs

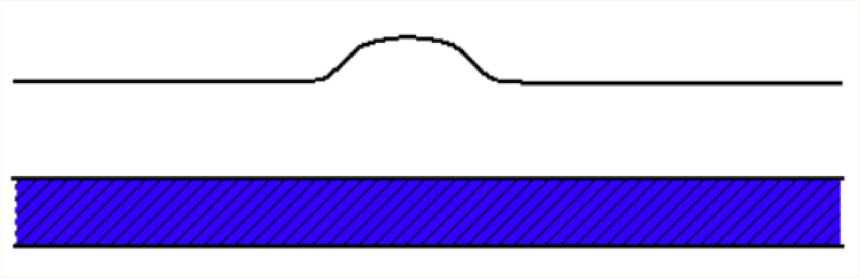




waveguide *piecewise string*

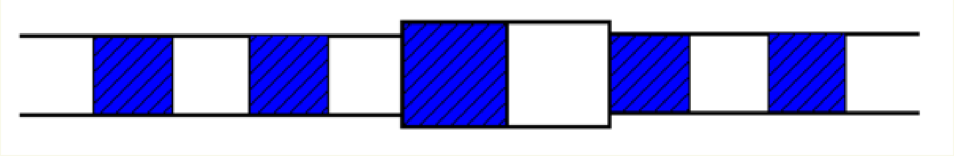
Dynamic homogenization

Localisation near thickness and cell resonance frequencies

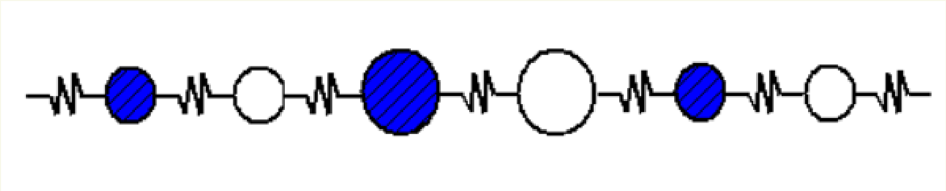


(*e.g. J.Kaplunov et al., 2005)*

continuous



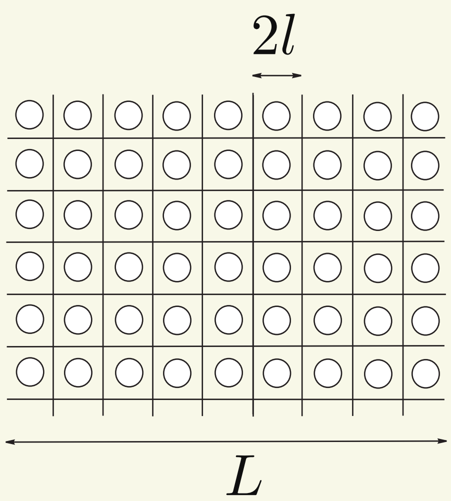
discrete



(*e.g. R.V.Craster et al., 2010,*
A.B. Movchan and L.I. Slepyan, 2014)

High frequency homogenization in 2D

(see R.V.Craster, J.Kaplunov & A.V.Pichugin, Proc R Soc A 2010)



$$\nabla_x \cdot [a(\mathbf{x})\nabla_x u(\mathbf{x})] + \omega^2 \rho(\mathbf{x}) u(\mathbf{x}) = 0$$

with double periodic $a(\mathbf{x})$ and $\rho(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)$

Small parameter $\epsilon = l/L \ll 1$

Scaling $\mathbf{X} = \dfrac{\mathbf{x}}{L}, \quad \boldsymbol{\xi} = \dfrac{\mathbf{x}}{l}$

Dynamic homogenization

Asymptotic series

$$u(\mathbf{X},\boldsymbol{\xi})=u_0(\mathbf{X},\boldsymbol{\xi})+\varepsilon u_1(\mathbf{X},\boldsymbol{\xi})+\varepsilon^2 u_2(\mathbf{X},\boldsymbol{\xi})+\dots$$

and

$$\lambda^2=\lambda_0^2+\varepsilon\lambda_1^2+\varepsilon^2\lambda_2^2+\dots, \qquad \text{where } \lambda=\frac{\omega l}{c_0}$$

Double periodicity - antiperiodicity conditions

$$\begin{aligned} u_i(\mathbf{X};-1,\xi_2)&=\pm u_i(\mathbf{X};1,\xi_2)\\ u_{i\xi_1}(\mathbf{X};-1,\xi_2)&=\pm u_{i\xi_1}(\mathbf{X};1,\xi_2)\\ u_i(\mathbf{X};\xi_1,-1)&=\pm u_i(\mathbf{X};\xi_1,1)\\ u_{i\xi_2}(\mathbf{X};\xi_1,-1)&=\pm u_{i\xi_2}(\mathbf{X};\xi_1,1)\end{aligned}$$

At leading order

$$u_0(\mathbf{X},\boldsymbol{\xi})=v_0(\mathbf{X})\,U_0(\boldsymbol{\xi})$$

and

$$\nabla_\xi\cdot[a(\boldsymbol{\xi})\nabla_\xi U_0]+\lambda_0^2c_0^2\rho(\boldsymbol{\xi})U_0=0 \qquad (*)$$

with the periodicity - antiperiodicity conditions on the cell contour

Dynamic homogenization

The final macroscale equation becomes

$$l^2 T_{ij} \frac{\partial^2 v_0}{\partial x_i \partial x_j} + \left(\lambda^2 - \lambda_0^2\right) v_0 = 0 \quad (i, j = 1, 2) \qquad (**)$$

with T_{ij} expressed through double integrals over the domain $-1 \leq \xi_1, \xi_2 \leq 1$ containing the double periodic eigenfunction $U_0(\boldsymbol{\xi})$ and a pair of single periodic functions $V_i(\boldsymbol{\xi})$, calculated from non-homogeneous boundary value problems for the equation (*).

Remarks.

(i) The equation (**) is valid near edges of stop bands.

(ii) The type of the equation (**) depends on problem parameters.

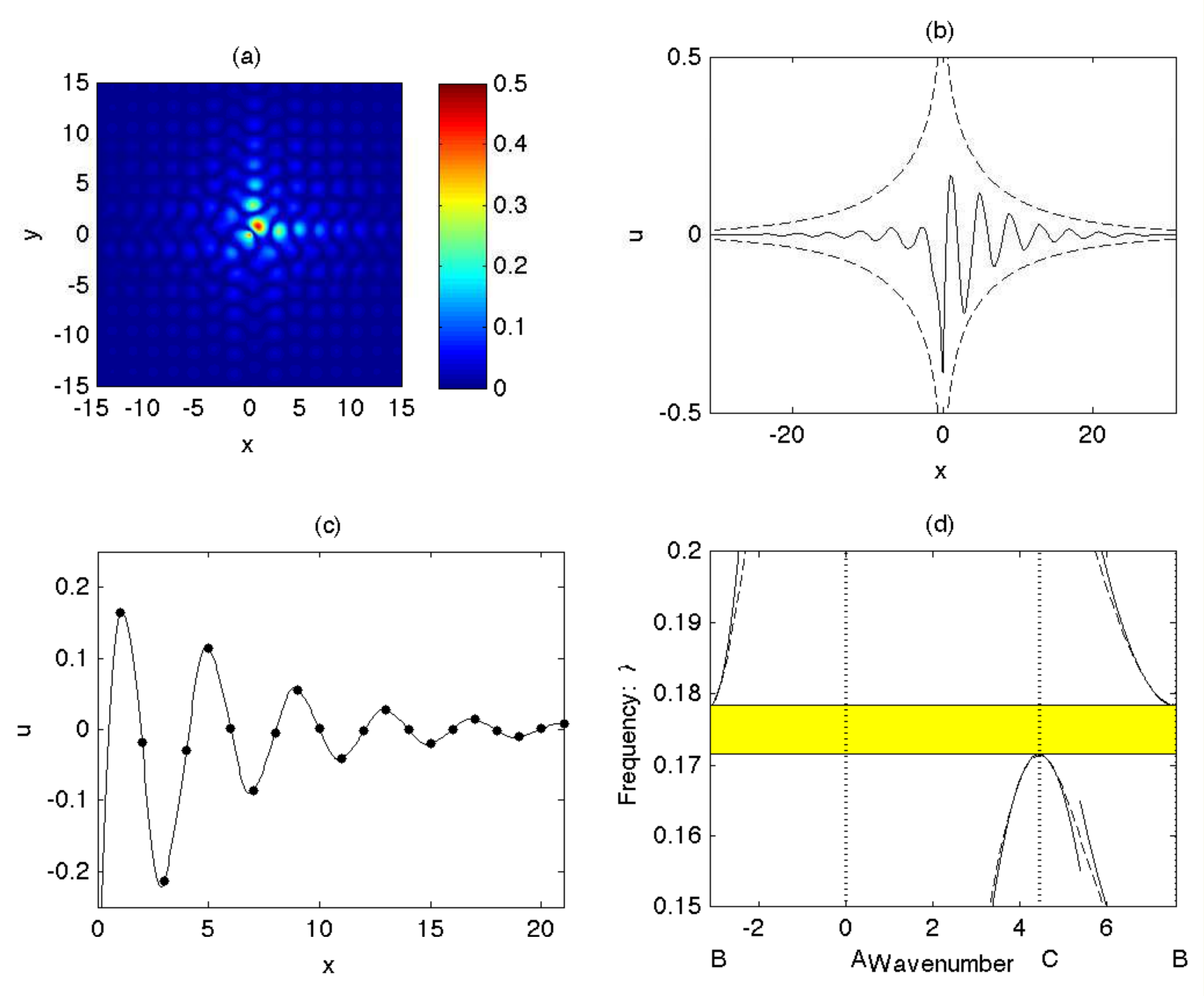
(iii) Simple explicit expressions for the coefficients T_{ij} are available only in the case of the checkerboard structures with piece-wise parameters governed by (see R.V.Craster, J.Kaplunov, E.Nolde & S.Guenneau, JOSA 2011)

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\omega^2}{c^2}\left[1 + g(x_1) + g(x_2)\right] u = 0$$

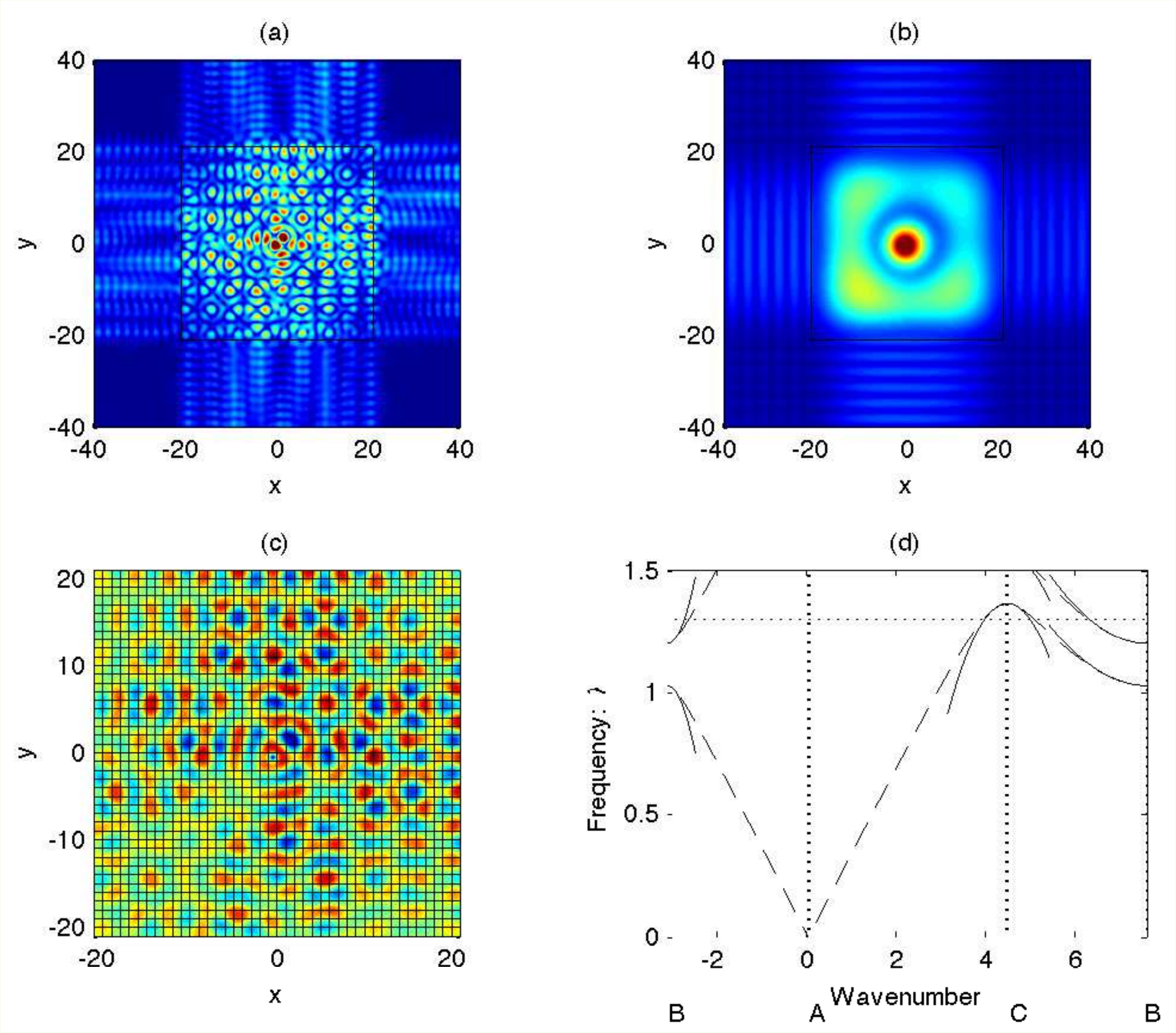
where $g_i(x_i) = 0$ for $-1 \leq x_i < 0$; $g_i(x_i) = r^2$ for $0 \leq x_i < 1$

Applications of high frequency homogenization theory
for checkerboards in optics

Defect modes

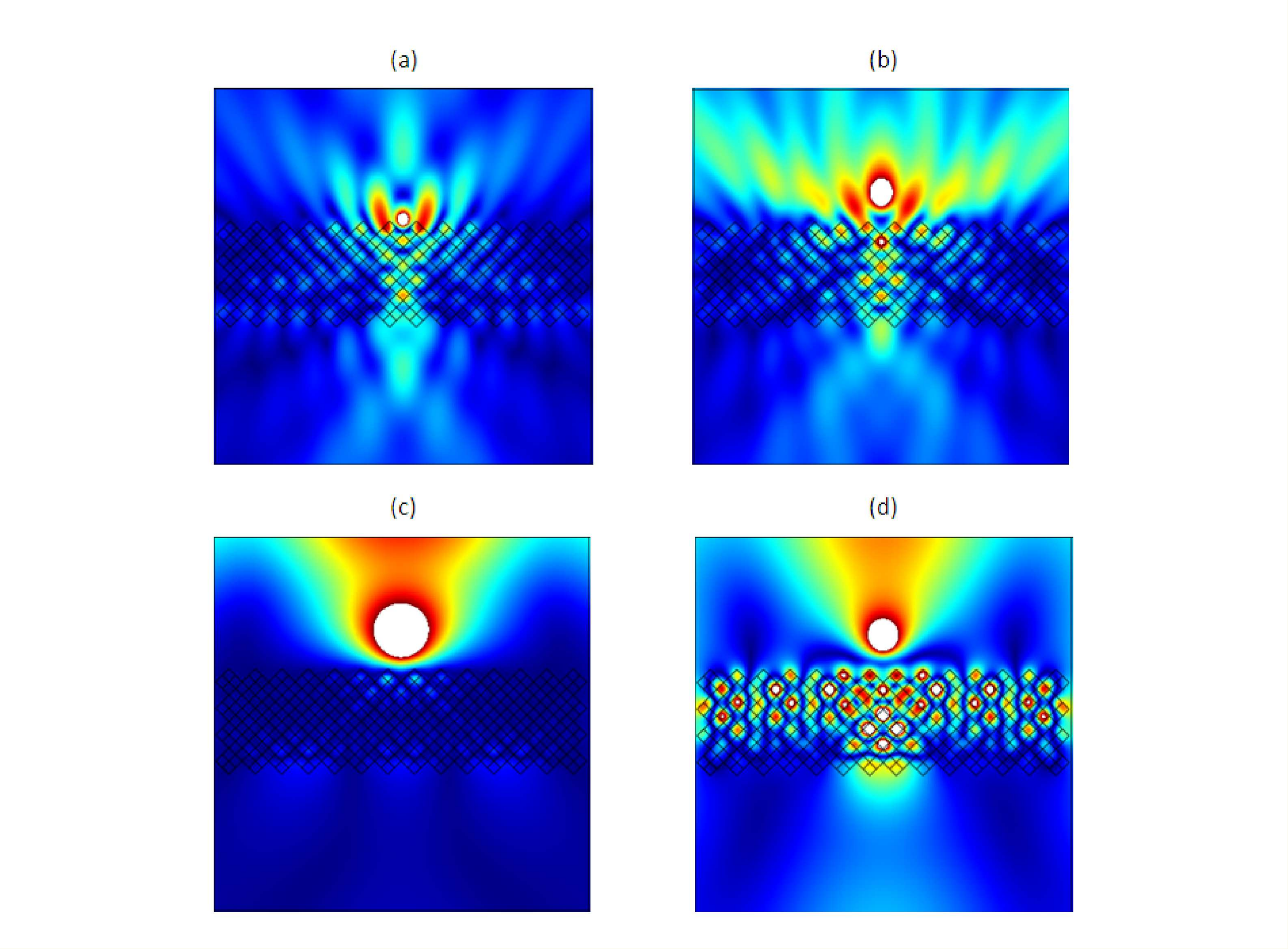


Ultra-refraction



Dynamic homogenization

All-angle negative refraction

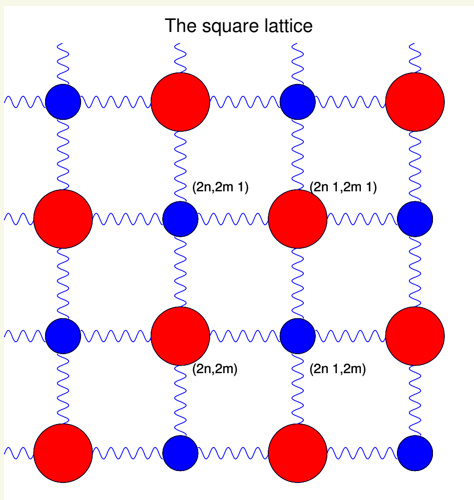


For further detail see *R.V.Craster, J.Kaplunov, E.Nolde & S.Guenneau, JOSA 2011*

High frequency homogenization for lattice structures

e.g. see *R.V.Craster, J.Kaplunov & J.Postnova, QJMAM 2010* for spring mass structures

and *E.Nolde, R.V.Craster & J.Kaplunov, JMPS 2011* for frame and truss structures



Two-scale approach

$$\mathbf{u} = \mathbf{u}\left(\mathbf{X}, \boldsymbol{\xi}\right)$$

continuous

discrete

$(\boldsymbol{\xi} = \{(0,0), (0,1), (1,0), (1,1)\})$

By applying Taylor series in $\mathbf{X}$ and periodicity - antiperiodicity conditions we arrive at a matrix-differential problem $[4 \times 4]$. It is

$$\Big[\underbrace{(A_0 - \lambda^2 M)}_{\textit{linear algebra}} + \varepsilon A_1(\partial_i, \lambda) + \varepsilon^2 A_2(\partial_i \partial_j, \lambda) + \ldots\Big]\,\mathbf{u}(\mathbf{X}, \boldsymbol{\xi}) = 0$$

with $\varepsilon = 1/N \ll 1, \quad \partial_i = \partial/\partial X_i, \quad M = \text{diag}\left(M_1, M_1, M_2, M_2\right)$

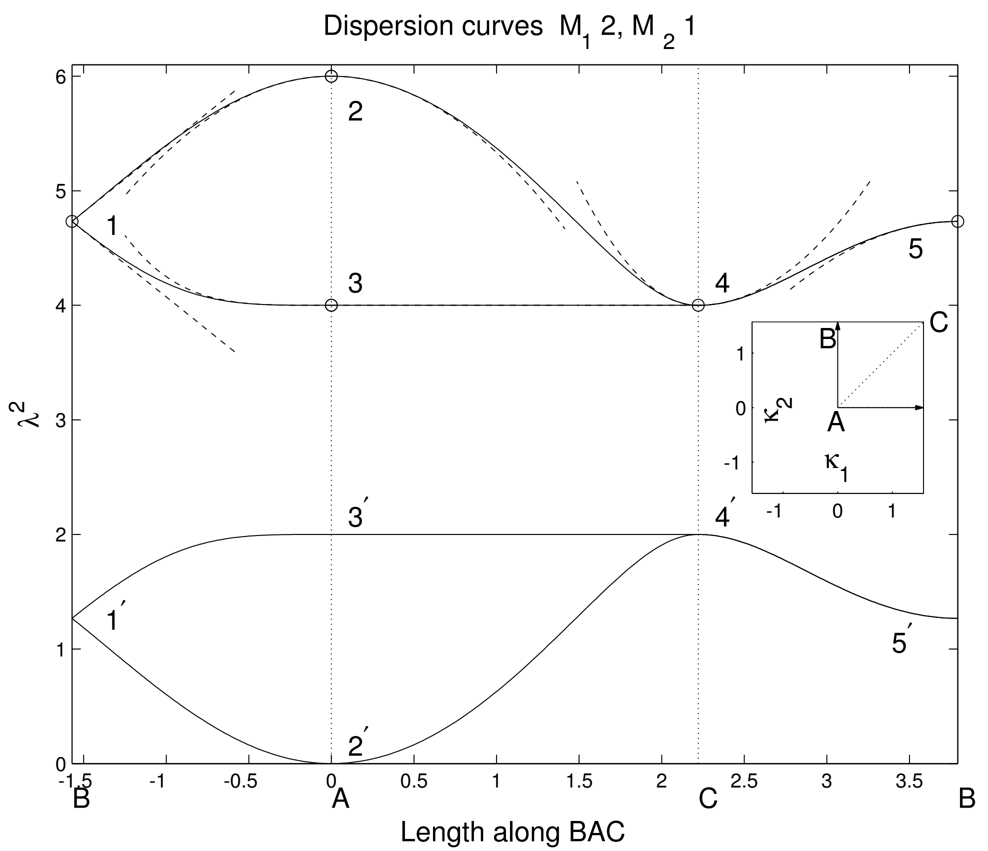
Variety of homogenized models

e.g. for periodicity in both directions when $\mathbf{u}(\mathbf{X}, \boldsymbol{\xi}) = v_0(\mathbf{X})\,[0, 0, -1, 1]^T$

The result is

$$\frac{l^4}{4(M_2 - M_1)}\left(\nabla_x^4 v_0 - 4\partial_{x_1}^2 \partial_{x_2}^2 v_0\right) + \left(\lambda^2 - \frac{4}{M_2}\right) v_0 = 0$$

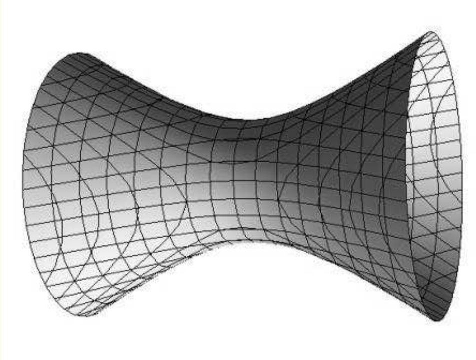
which is particularly handy for analyzing localized phenomena.



Dynamic homogenization

Dynamic anisotropy

- Similarity with hyperbolic shells



- Elastic lattices, e.g. see D.J. Colquitt et al., 2012

Work in progress

- Non-linear high frequency waves in periodic media
- Justification/refinement of *ad-hoc* micropolar theories and shear deformation plate and shell theories, e.g. Cosserat vs Timoshenko-Reissner-Mindlin