Low-frequency vibrations of strongly inhomogeneous layered elastic plates

L.A. Prikazchikova

School of Computing and Mathematics,
Keele University,
UK

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- Out-of-plane problem for asymmetric plate
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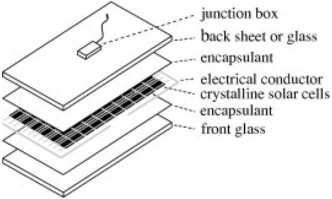
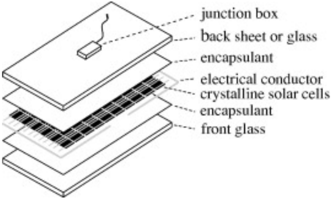
High-contrast layered structures

- Photovoltaic panels ($\mu_c/\mu_s \sim 10^{-2} - 10^{-5}$)

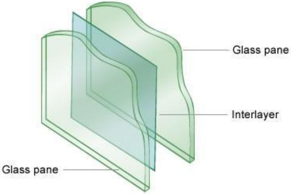


High-contrast layered structures

- Photovoltaic panels ($\mu_c/\mu_s \sim 10^{-2} - 10^{-5}$)

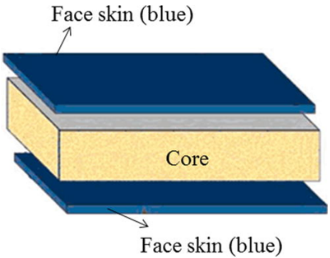


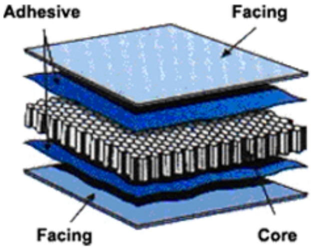
- Laminated glass



Sandwich structures

- Classical sandwich plate





Preliminary remarks

Consider bending of homogeneous isotropic plate of width 2h

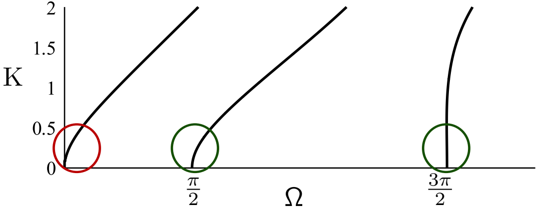
Rayleigh-Lamb dispersion equation

$$\gamma^4 \frac{\sinh \alpha}{\alpha} \cosh \beta - \beta^2 \mathrm{K}^2 \cosh \alpha \frac{\sinh \beta}{\beta} = 0, \quad \text{where}$$

$$\alpha^2 = \mathrm{K}^2 - \varkappa^2 \Omega^2, \quad \beta^2 = \mathrm{K}^2 - \Omega^2, \quad \gamma^2 = \mathrm{K}^2 - \frac{1}{2}\Omega^2, \quad \varkappa = \frac{c_2}{c_1}$$

Scaled wave number and frequency

$$\mathrm{K} = \mathrm{kh}, \quad \Omega = \frac{\omega \mathrm{h}}{c_2}$$



Long wave approximations ($K \ll 1$)

◯ Low-frequency ($\Omega \ll 1$)

At the leading order $\Omega \sim K^2$ or

$$D_a \frac{d^4 w}{d\xi^4} - \Omega^2 w = 0, \qquad \xi = \frac{x}{h}$$

Kirchhoff equation $\equiv$ leading order approximation

◯ High-frequency approximations near cut-off frequencies $\Omega_* \sim 1$ ($|\Omega - \Omega_*| \ll 1$)

At the leading order $K \sim |\Omega^2 - \Omega_*^2|^{\frac{1}{2}}$

$$P_a \frac{d^2 w}{d\xi^2} + (\Omega^2 - \Omega_*^2) w = 0$$

No overlap frequency regions $\Rightarrow$

No chance for 2-mode uniform approximations!

J. Kaplunov et al, Dynamics of thin walled elastic bodies,1998

Composite (non-uniformly asymptotic) plate theories

Originate from Timoshenko-Reissner-Mindlin ad hoc theories.

$$\overbrace{\mathrm{D_a}\frac{\mathrm{d}^4\mathrm{W}}{\mathrm{d}\xi^4} - \Omega^2\mathrm{W}}^{\text{low-frequency}} \qquad \mathrm{D_a}\frac{\mathrm{d}^4\mathrm{W}}{\mathrm{d}\xi^4} - \underbrace{\Omega^2\mathrm{W} + \mathrm{B_a}\Omega^2\frac{\mathrm{d}^2\mathrm{W}}{\mathrm{d}\xi^2} + \mathrm{C_a}\Omega^4\mathrm{W}}_{\text{high-frequency}} = 0,$$

$\mathrm{B_a}, \mathrm{C_a}, \mathrm{D_a}$ - constants

V.L. Berdichevsky. Variational principles of continuum mechanics: I. Fundamentals, 2009

K.C. Le. Vibrations of shells and rods, 2012

I.V. Andrianov et al,Asymptotical mechanics of thin-walled structures, 2013

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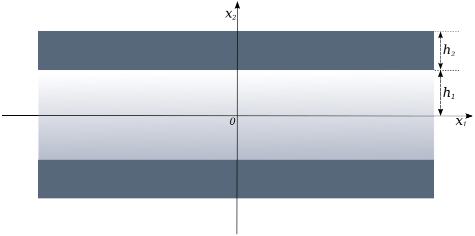
Anti-plane scalar problem

Anti-plane scalar problem for asymmetric plates

In-plane vibrations of three-layered plates

Statement of the problem



Equations of motion

$$\sigma^{\mathrm{q}}_{\mathrm{ji,j}} = \rho_{\mathrm{q}} \ddot{\mathrm{u}}^{\mathrm{q}}_{\mathrm{i}}, \quad \mathrm{q} = \mathrm{c}, \mathrm{s} \quad \text{for core and skin layers}$$

Boundary and continuity conditions

$$\sigma^{\mathrm{s}}_{12} = 0, \quad \sigma^{\mathrm{s}}_{22} = 0 \quad \text{at} \quad \mathrm{x}_2 = \mathrm{h_c} + \mathrm{h_s}$$

$$\sigma^{\mathrm{c}}_{12} = \sigma^{\mathrm{s}}_{12}, \quad \sigma^{\mathrm{c}}_{22} = \sigma^{\mathrm{s}}_{22} \quad \text{and} \quad \mathrm{u}^{\mathrm{c}}_1 = \mathrm{u}^{\mathrm{s}}_1, \quad \mathrm{u}^{\mathrm{c}}_2 = \mathrm{u}^{\mathrm{s}}_2 \quad \text{at} \quad \mathrm{x}_2 = \mathrm{h_c}$$

Dispersion relation for antisymmetric motion

$$
\begin{aligned}
&4\mathrm{K}^2\mathrm{h}^3\alpha_2\beta_2\mathrm{F}_4\,[\mathrm{F}_1\mathrm{F}_2\mathrm{C}_{\beta_1}\mathrm{S}_{\alpha_1} - 2\alpha_1\beta_1(\mu-1)\mathrm{F}_3\mathrm{C}_{\alpha_1}\mathrm{S}_{\beta_1}] \\
&+ \mathrm{h}\alpha_2\beta_2\mathrm{C}_{\alpha_2}\mathrm{C}_{\beta_2}\,\big[4\alpha_1\beta_1\mathrm{K}^2\left(\mathrm{h}^4\mathrm{F}_3^2 + {\mathrm{F}_4}^2(\mu-1)^2\right)\mathrm{C}_{\alpha_1}\mathrm{S}_{\beta_1} \\
&\qquad\qquad\qquad\qquad - \left(4\mathrm{K}^4\mathrm{h}^4\mathrm{F}_2^2 + {\mathrm{F}_4}^2\mathrm{F}_1^2\right)\mathrm{S}_{\alpha_1}\mathrm{C}_{\beta_1}\big] \\
&+ \mathrm{C}_{\beta_2}\mathrm{S}_{\alpha_2}\mu\beta_2(\beta_2^2 - \mathrm{K}^2\mathrm{h}^2)(\beta_1^2 - \mathrm{K}^2)\,\big[4\alpha_2^2\beta_1\mathrm{K}^2\mathrm{h}^2\mathrm{S}_{\alpha_1}\mathrm{S}_{\beta_1} - {\mathrm{F}_4}^2\alpha_1\mathrm{C}_{\alpha_1}\mathrm{C}_{\beta_1} \\
&+ \mathrm{C}_{\alpha_2}\mathrm{S}_{\beta_2}\mu\alpha_2(\beta_2^2 - \mathrm{K}^2\mathrm{h}^2)(\beta_1^2 - \mathrm{K}^2)\,\big[4\alpha_1\beta_2^2\mathrm{K}^2\mathrm{h}^2\mathrm{C}_{\alpha_1}\mathrm{C}_{\beta_1} - {\mathrm{F}_4}^2\beta_1\mathrm{S}_{\alpha_1}\mathrm{S}_{\beta_1} \\
&+ \mathrm{h}^3\mathrm{S}_{\alpha_2}\mathrm{S}_{\beta_2}\,\big[\left(4\alpha_2^2\beta_2^2\mathrm{K}^2\mathrm{F}_1^2 + \mathrm{K}^2{\mathrm{F}_4}^2\mathrm{F}_2^2\right)\mathrm{C}_{\beta_1}\mathrm{S}_{\alpha_1} \\
&\qquad\qquad -\alpha_1\beta_1\left(16\alpha_2^2\beta_2^2(\mu-1)^2\mathrm{K}^4 + {\mathrm{F}_4}^2\mathrm{F}_3^2\right)\mathrm{C}_{\alpha_1}\mathrm{S}_{\beta_1}\big] = 0
\end{aligned}
$$

Non-dimensional scaled frequency and wave number

$$\Omega = \frac{\omega \mathrm{h_c}}{\mathrm{c}_2^\mathrm{c}}, \quad \mathrm{K} = \mathrm{kh_c}$$

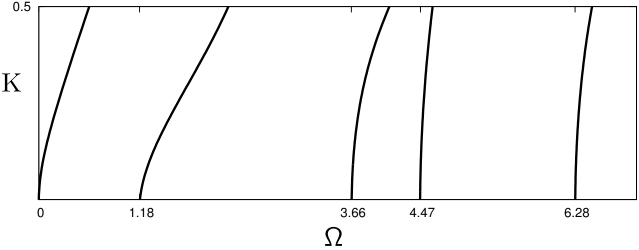
$\mathrm{F_i}, \quad \mathrm{i} = 1..4, \qquad \alpha_\mathrm{j}, \beta_\mathrm{j}, \quad \mathrm{j} = 1,2$ - functions of Ω and K,

$\mathrm{C}_{\alpha_\mathrm{j}}, \mathrm{C}_{\beta_\mathrm{j}}, \mathrm{S}_{\alpha_\mathrm{j}}, \mathrm{S}_{\beta_\mathrm{j}}$ - hyperbolic functions

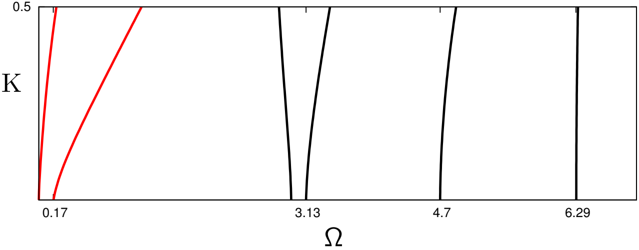
P. C. Y. Lee et al, Journal of Elasticity, 1979.

Dispersion curves

No contrast



Effect of contrast



1D eigenvalue problem for shear cut-off (antisymmetric motion)

Setting $\partial u_1^q / \partial x_1 = u_2^q = 0$ in the above problem we have $(u_1^q = u_1^q(x_2))$

Equations of motion

$$\frac{\mathsf{d}^2 u_1^q}{\mathsf{d}\, x_2^2} + \frac{\omega^2}{c_2^q} u_1^q = 0, \qquad q = c, s,$$

subject to the boundary and continuity conditions

$$\frac{\mathsf{d}\, u_1^s}{\mathsf{d}\, x_2} = 0 \quad \text{at} \quad x_2 = \pm(h_c + h_s),$$

$$\mu_c \frac{\mathsf{d}\, u_1^c}{\mathsf{d}\, x_2} = \mu_s \frac{\mathsf{d}\, u_1^s}{\mathsf{d}\, x_2}, \quad u_1^c = u_1^s \quad \text{at} \quad x_2 = \pm h_c.$$

Cut-off shear frequencies

Equation for cut-off shear frequencies:

$$\tan(\Omega)\tan\left(\sqrt{\frac{\mu}{\rho}}\mathrm{h}\ \Omega\right)=\sqrt{\mu\rho}$$

Condition for a first shear cut-off frequency to be small

$$\rho \ll \mathrm{h} \ll \mu^{-1}, \quad \Omega \approx \left(\frac{\rho}{\mathrm{h}}\right)^{1/2},$$

where

$$\mu=\frac{\mu_{\mathrm{c}}}{\mu_{\mathrm{s}}}, \quad \rho=\frac{\rho_{\mathrm{c}}}{\rho_{\mathrm{s}}}, \qquad \mathrm{h}=\frac{\mathrm{h}_{\mathrm{s}}}{\mathrm{h}_{\mathrm{c}}}.$$

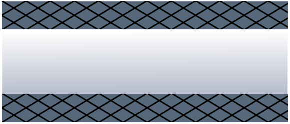
J. Kaplunov et al, Journal of Sound and Vibration,2016.

Practical examples ($\rho \ll \mathrm{h} \ll \mu^{-1}$)

A. Photovoltaic panels

$\mu \ll 1, \mathrm{h} \sim 1, \rho \sim \mu$

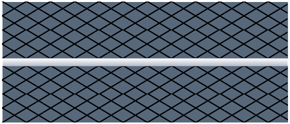
stiff skin layers and light core layer



B. Laminated glass

$\mu \ll 1, \mathrm{h} \sim \mu^{-1/2}, \rho \sim \mu$

stiff skin layers and light thin core layer



C. Sandwich structures

$\mu \ll 1, \mathrm{h} \sim \mu, \rho \sim \mu^2$

stiff thin skin layers and light core layer

Unusually low first shear cut-off frequencies!

Long-wave low-frequency asymptotic approximation

For $\mathrm{K} \ll 1$ and $\Omega \ll 1$

$$\gamma_1\Omega^2 + \gamma_2\mathrm{K}^4 + \gamma_3\mathrm{K}^2\Omega^2 + \gamma_4\mathrm{K}^6 + \gamma_5\Omega^4 + \gamma_6\mathrm{K}^4\Omega^2 + \gamma_7\mathrm{K}^8 + \gamma_8\mathrm{K}^2\Omega^4 + \gamma_9\mathrm{K}^2\Omega^6 + \gamma_{10}\Omega^6 + ... = 0$$

Multi-parametric analysis

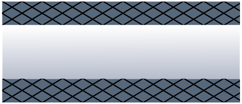
$$\mu \ll 1, \quad \mathrm{h} \sim \mu^{\mathrm{a}}, \quad \rho \sim \mu^{\mathrm{b}}$$

Expanding coefficients

$$\gamma_{\mathrm{i}} \to \mathrm{G_i}\mu^{\mathrm{c}}, \qquad \mathrm{G_i} \sim 1$$

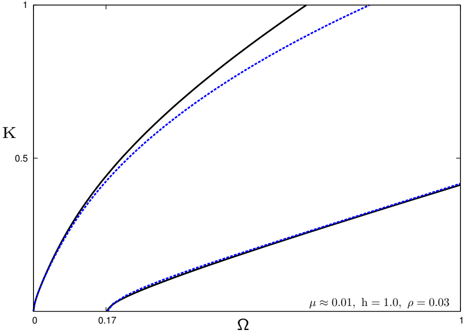
A. Photovoltaic panels

Plate with stiff outer layers and light core
$\mu \ll 1, \quad \mathrm{h} \sim 1, \quad \rho \sim \mu$



Retain leading order terms for both modes:

1. fundamental mode ($\Omega \sim \mathrm{K}^2$)
2. shear mode with cut-off $\Omega_{\mathrm{sh}} \sim \sqrt{\mu}$

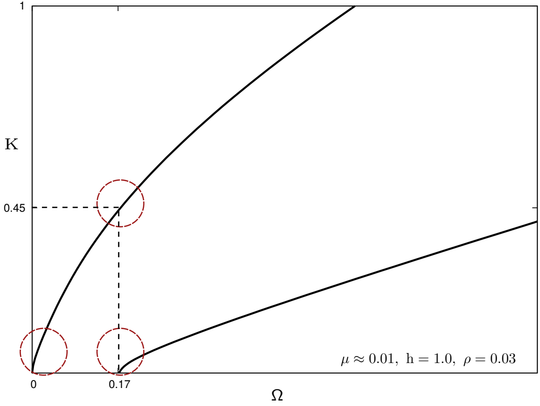


Five term two-mode approximation

$$\mathrm{G}_1\mu\Omega^2 + \mathrm{G}_2\mu\mathrm{K}^4 + \mathrm{G}_3\mathrm{K}^2\Omega^2 + \mathrm{G}_4\mathrm{K}^6 + \mathrm{G}_5\Omega^4 = 0$$

Local approximations

Three local approximations can be obtained from the two-mode approximation



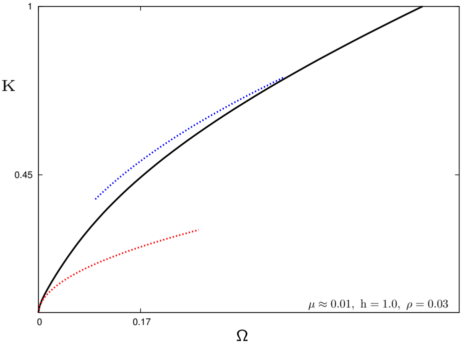
Local approximations for the fundamental mode

In the vicinity of zero frequency

$$G_1\Omega^2 + G_2K^4 = 0, \quad 0 < K \ll \sqrt{\mu}, \quad \Omega \ll \mu$$

At higher frequencies, including the vicinity of shear cut-off $\Omega_{sh} \sim \sqrt{\mu}$

$$G_3\Omega^2 + G_4K^4 = 0, \quad \sqrt{\mu} \ll K \ll 1, \quad \mu \ll \Omega \ll 1$$

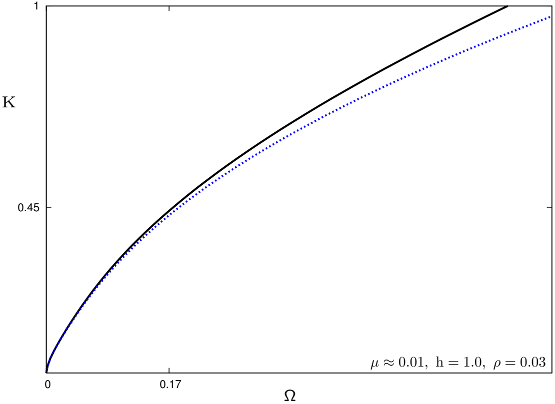


Kirchhoff theory already does not work at $\Omega \sim \mu$!

Uniform approximation for the fundamental mode

Taking both local approximations we derive a uniform one:

$$G_1\mu\Omega^2 + G_2\mu K^4 + G_3 K^2\Omega^2 + G_4 K^6 = 0$$

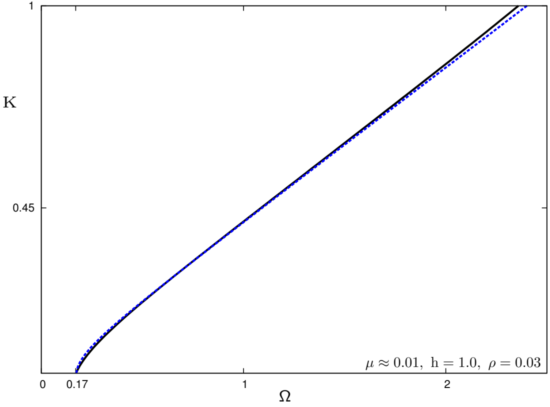


Also valid in the transition region $\Omega \sim \mu, K \sim \sqrt{\mu}$

Near cut-off approximation

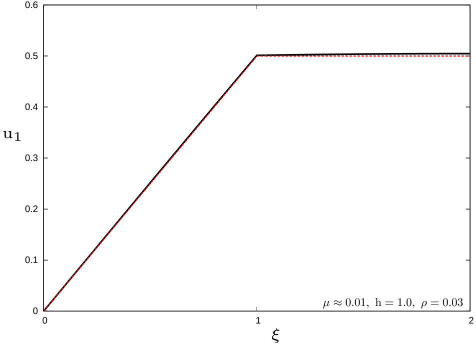
For $\Omega \sim \sqrt{\mu}$, $\mathrm{K} \ll 1$

$$\mathrm{G}_1\mu + \mathrm{G}_3\mathrm{K}^2 + \mathrm{G}_5\Omega^2 = 0$$



Displacement u_1 near shear cut-off frequency

Horizontal displacement u_1 and approximation for $\mu \approx 0.01$ in shear mode ($K = 0$)



At the leading order

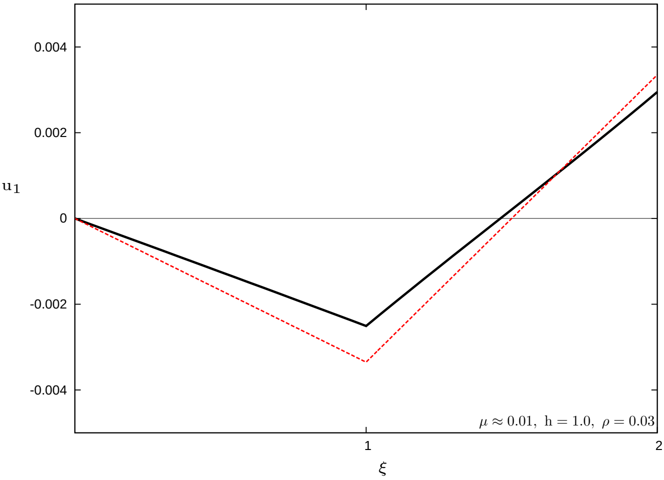
$$u_1 = \begin{cases} \frac{\xi}{1+h}, & \text{for } \xi = 0..1 \\ \frac{1}{1+h}, & \text{for } \xi = 1..1+h \end{cases}$$

where

$$\xi = \frac{x_1}{h_c}$$

Displacement u_1 near shear cut-off frequency

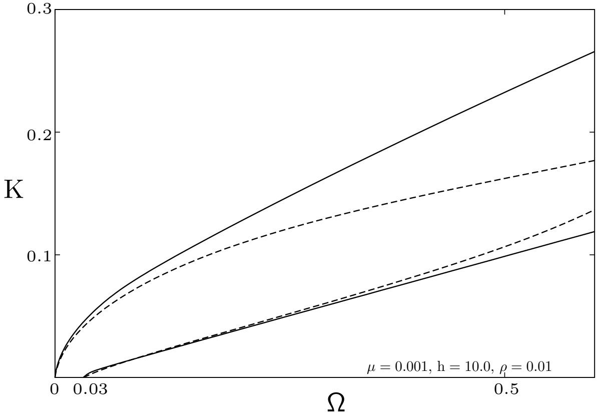
Displacement u_1 and approximation for $\mu \approx 0.01$ on fundamental mode (at $K = 0.45$)



B. Laminated glass. Two-mode approximation

$\Omega_{\mathrm{sh}} \sim \mu^{1/2}$ leads to the two-mode uniform approximation

$$\mathrm{G}_1\mu^3\Omega^2 + \mathrm{G}_2\mu^{3/2}\mathrm{K}^4 + \mathrm{G}_3\mu^{3/2}\mathrm{K}^2\Omega^2 + \mathrm{G}_4\mathrm{K}^6 + \mathrm{G}_5\mu^2\Omega^4 = 0,$$



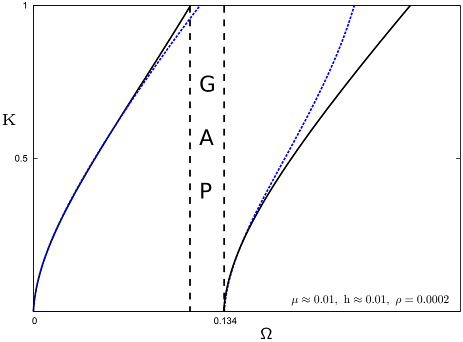
C. Sandwich structure. Two-mode approximation

Plate with stiff outer layers and light core

$\mu \ll 1, \quad \mathrm{h} \sim \mu, \quad \rho \sim \mu^2$

Local approximation:

fundamental mode ($\Omega \ll \sqrt{\mu}$) and shear mode ($\Omega_{\mathrm{sh}} \sim \sqrt{\mu}$)



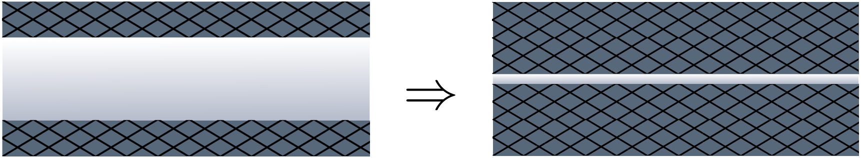
Two-mode approximation

$$\mathrm{G}_1\mu\Omega^2 + \mathrm{G}_2\mu^2\mathrm{K}^4 + \mu\mathrm{K}^2\Omega^2\left(\mathrm{G}_3 + \frac{\rho_\mu}{\mathrm{h}_\mu}\mathrm{G}_8\right) + \mathrm{G}_5\Omega^4 = 0$$

Composite non-uniform approximations!

Transition from a uniform to non-uniform approximation

Where is transition from uniform approximation to a composite (non-uniform) one?

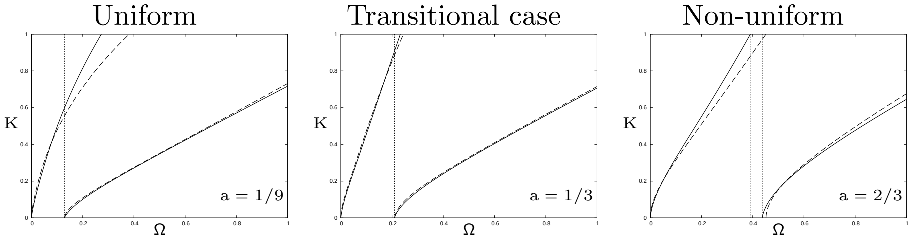


$$\rho = \frac{\rho_c}{\rho_s} \ll 1, \quad \mu = \frac{\mu_c}{\mu_s} \sim \rho, \quad h = \frac{h_s}{h_c} \sim \rho^a, \quad 0 \leqslant a < 1$$

Small thickness shear cut-off frequency

$$\Omega_{sh} \approx \left(\frac{\rho}{h}\right)^{1/2} \sim \rho^{(1-a)/2} \ll 1$$

Transition from a uniform to non-uniform approximation



- Uniform $0 \leq a \leq 1/3$

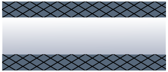
$$\rho^{1-a}G_1\Omega^2 + \rho^{1-a}G_2K^4 + G_3K^2\Omega^2 + \frac{1}{3}\rho^{2a}G_2K^6 + G_5\Omega^4 = 0.$$

- Non-uniform $1/3 < a < 1$

$$\rho^{1-a}G_1\Omega^2 + \rho^{1-a}G_2K^4 + G_3K^2\Omega^2 + G_5\Omega^4 = 0.$$

In progress: 2D PDEs for strongly inhomogeneous plates

Uniformly asymptotic



$$G_1 \mu u_{tt} + G_2 \mu \Delta^2 u + G_3 \Delta u_{tt} + G_4 \Delta^3 u + G_5 u_{tttt} = 0$$

Composite

$$G_1 \mu u_{tt} + G_2 \mu^2 \Delta^2 u + G_3 \mu \Delta u_{tt} + G_5 u_{tttt} + G_8 \Delta u_{tttt} = 0$$

Not easy to justify!

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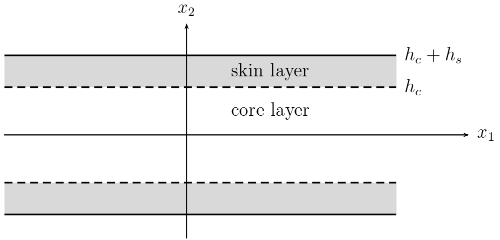
Anti-plane antisymmetric motion



Equations of motion

$$\frac{\partial \sigma_{13}^{\mathrm{q}}}{\partial \mathrm{x}_1} + \frac{\partial \sigma_{23}^{\mathrm{q}}}{\partial \mathrm{x}_2} - \rho_{\mathrm{q}} \frac{\partial^2 \mathrm{u}_{\mathrm{q}}}{\partial \mathrm{t}^2} = 0, \qquad \mathrm{q} = \mathrm{c}, \mathrm{s},$$

with

$$\sigma_{\mathrm{i}3}^{\mathrm{q}} = \mu_{\mathrm{q}} \frac{\partial \mathrm{u}_{\mathrm{q}}}{\partial \mathrm{x}_{\mathrm{i}}}, \quad \mathrm{i} = 1, 2,$$

u_{q} are out of plane displacements, $\sigma_{\mathrm{i}3}^{\mathrm{q}}$ are shear stresses.

Dispersion relation

Continuity conditions along interfaces $x_2 = \pm h_c$

$$\sigma_{23}^{c} = \sigma_{23}^{s} \quad \text{and} \quad u_c = u_s.$$

Traction-free boundary conditions

$$\sigma_{23}^{s} = 0 \qquad \text{at} \qquad x_2 = \pm(h_c + h_s).$$

Equations of motion

$$\Delta u_q - \frac{1}{(c_2^q)^2}\frac{\partial^2 u_q}{\partial t^2} = 0, \quad q = c, s.$$

Dispersion relation

$$\mu\alpha_1 \cosh(\alpha_1)\cosh(\alpha_2 h) + \alpha_2 \sinh(\alpha_1)\sinh(\alpha_2 h) = 0,$$

with

$$\alpha_1 = \sqrt{K^2 - \Omega^2}, \qquad \alpha_2 = \sqrt{K^2 - \frac{\mu}{\rho}\Omega^2},$$

$$\Omega = \frac{\omega h_c}{c_2^c}, \quad K = k h_c, \quad h = \frac{h_s}{h_c}, \quad \mu = \frac{\mu_c}{\mu_s}, \quad \rho = \frac{\rho_c}{\rho_s}.$$

Exact solutions for displacements and stresses

$$\mathrm{u_c} = \mathrm{h_c}\frac{\sinh(\alpha_1\xi_{2c})}{\alpha_1}, \quad \sigma_{13}^{c} = \mathrm{i}\mu_c \mathrm{K}\frac{\sinh(\alpha_1\xi_{2c})}{\alpha_1}, \quad \sigma_{23}^{c} = \mu_c \cosh(\alpha_1\xi_{2c}),$$

and

$$\mathrm{u_s} = \mathrm{h_c}\beta\left(\cosh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right] - \tanh\left[\alpha_2(\mathrm{h}+1)\right]\sinh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right]\right),$$
$$\sigma_{13}^{s} = \mathrm{i}\mu_s \mathrm{K}\beta\left(\cosh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right] - \tanh\left[\alpha_2(\mathrm{h}+1)\right]\sinh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right]\right),$$
$$\sigma_{23}^{s} = \mu_s\alpha_2\beta\left(\sinh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right] - \tanh\left[\alpha_2(\mathrm{h}+1)\right]\cosh\left[\alpha_2(\mathrm{h}\xi_{2s}+1)\right]\right),$$

where

$$\beta = \frac{\sinh\alpha_1}{\alpha_1\big(\cosh\alpha_2 - \sinh\alpha_2\tanh[\alpha_2(\mathrm{h}+1)]\big)}.$$

Dimensionless variables

$$\xi_{2c} = \frac{\mathrm{x}_2}{\mathrm{h_c}}, \qquad 0 \le \mathrm{x}_2 \le \mathrm{h_c},$$
$$\xi_{2s} = \frac{\mathrm{x}_2 - \mathrm{h_c}}{\mathrm{h_s}}, \quad \mathrm{h_c} \le \mathrm{x}_2 \le \mathrm{h_c} + \mathrm{h_s}.$$

Long-wave low-frequency limit

Polynomial dispersion relation

$$\mu + \gamma_1 \mathrm{K}^2 + \gamma_2 \mathrm{K}^4 + \gamma_3 \mathrm{K}^2 \Omega^2 + \gamma_4 \Omega^2 + \gamma_5 \Omega^4 + \cdots = 0,$$

with

$$\gamma_1 = \frac{\mu}{2}\left(1 + \mathrm{h}^2\right) + \mathrm{h},$$

$$\gamma_2 = \frac{\mu}{24}\left(1 + 6\mathrm{h}^2 + \mathrm{h}^4\right) + \frac{\mathrm{h}}{6}(1 + \mathrm{h}^2),$$

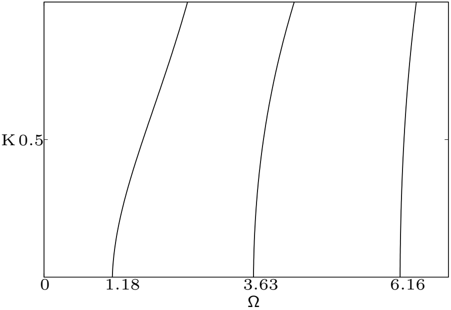
$$\gamma_3 = -\frac{\mu}{12}(1 + 3\mathrm{h}^2) - \frac{\mathrm{h}}{6} - \frac{\mu \mathrm{h}}{12\rho}(2 + 3\mu \mathrm{h}) - \frac{\mu \mathrm{h}^3}{12\rho}(4 + \mu \mathrm{h}),$$

$$\gamma_4 = -\frac{\mu}{2} - \frac{\mu \mathrm{h}}{\rho}\left(1 + \frac{\mu \mathrm{h}}{2}\right),$$

$$\gamma_5 = \frac{\mu}{24} + \frac{\mu \mathrm{h}}{12\rho}(2 + 3\mu \mathrm{h}) + \frac{\mu^2 \mathrm{h}^3}{24\rho^2}(4 + \mu \mathrm{h}).$$

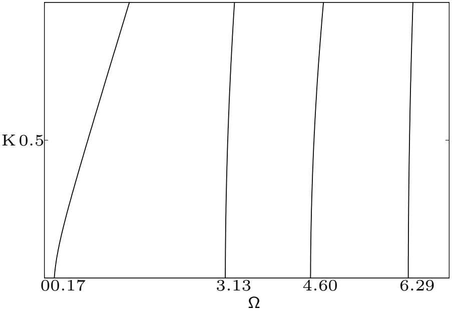
Dispersion curves

No contrast



$\mu = 0.232,\ \rho = 3.0,\ h = 1.0$

Effect of contrast



$\mu = 0.014,\ \rho = 0.03,\ h = 1.0$

- No fundamental mode. It appears in case of symmetric motion.
- The lowest cut-off frequency in case of a contrast is $\Omega = 0.17$

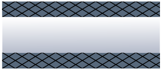
Consider two setups of the contrast:

A. Photovoltaic panels and B. Sandwich structures

A. Photovoltaic panels. Shortened polynomial dispersion relation

Plate with stiff outer layers and light core

$\mu \ll 1, \quad \mathrm{h} \sim 1, \quad \rho \sim \mu$



$$\gamma_1 \sim \gamma_2 \sim \gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$$

Shortened dispersion relation

$$\frac{\mu}{\mathrm{h}} + \mathrm{K}^2 - \frac{1}{\rho_\mu}\Omega^2 = 0.$$

Scaled dimensionless frequency and wavenumber

$$\Omega^2 = \mu^\alpha \Omega_*^2 \quad \text{and} \quad \mathrm{K}^2 = \mu^\alpha \mathrm{K}_*^2,$$

where $\Omega_* \sim \mathrm{K}_* \sim 1$ and $0 < \alpha \leq 1$.

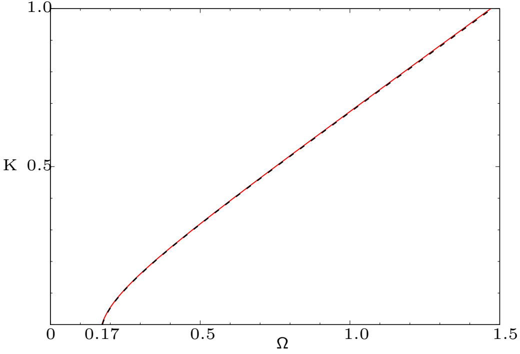
α covers the whole long-wave low-frequency band, given by $\Omega \ll 1$, and $\mathrm{K} \ll 1$.

Shortened polynomial dispersion relation

Dispersion relation expressed in Ω_* and K_* becomes

$$\Omega_*^2 = \rho_\mu \left(\mathrm{K}_*^2 + \frac{\mu^{1-\alpha}}{\mathrm{h}} \right).$$

At $\alpha < 1$ we have $\Omega_* \sim \sqrt{\rho_\mu}\mathrm{K}_*$ or $\omega \sim c_2^s k$, corresponding to the short-wave limit for stiffer skin layers.



B. Sandwich structure. Shortened polynomial dispersion relation

Plate with stiff outer layers and light core

$\mu \ll 1, \quad \mathrm{h} \sim \mu, \quad \rho \sim \mu^2$

$$\gamma_1 \sim \gamma_2 \sim \mu \quad \text{and} \quad \gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$$

Approximate dispersion relation

$$\mu + \mu \left(\frac{1}{2} + \mathrm{h}_\mu \right) \mathrm{K}^2 - \frac{\mathrm{h}_\mu}{6\rho_\mu} \mathrm{K}^2 \Omega^2 - \left(\frac{\mu}{2} + \frac{\mathrm{h}_\mu}{\rho_\mu} \right) \Omega^2 + \frac{\mathrm{h}_\mu}{6\rho_\mu} \Omega^4 = 0.$$

Normalized wavenumber and frequency

$$\mathrm{K}^2 = \mu \mathrm{K}_*^2 \quad \text{and} \quad \Omega^2 = \mu \Omega_*^2,$$

we obtain

$$1 + \mu \left(\frac{1}{2} + \mathrm{h}_\mu \right) \mathrm{K}_*^2 - \mu \frac{\mathrm{h}_\mu}{6\rho_\mu} \mathrm{K}_*^2 \Omega_*^2 - \left(\frac{\mu}{2} + \frac{\mathrm{h}_\mu}{\rho_\mu} \right) \Omega_*^2 + \mu \frac{\mathrm{h}_\mu}{6\rho_\mu} \Omega_*^4 = 0.$$

Shortened polynomial dispersion relation

Adapt a near cut-off asymptotic expansion in the form

$$\Omega_*^2 = \Omega_0^2 + \mu\Omega_1^2 + \cdots$$

where

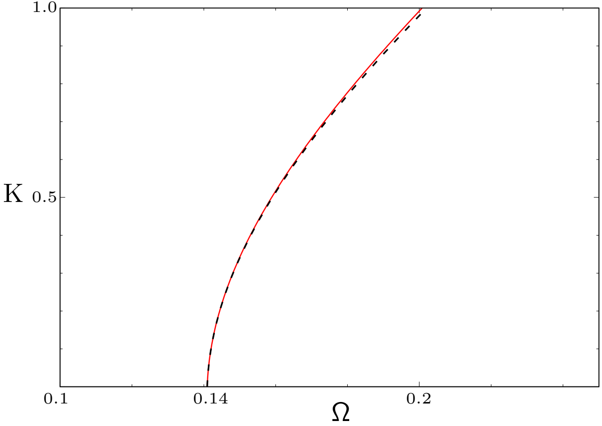
$$\Omega_0^2 = \frac{\rho_\mu}{\mathrm{h}_\mu} \quad \text{and} \quad \Omega_1^2 = \frac{\rho_\mu}{\mathrm{h}_\mu}\left(\frac{1}{3} + \mathrm{h}_\mu\right)\mathrm{K}_*^2 - \frac{1}{3}\frac{\rho_\mu^2}{\mathrm{h}_\mu^2},$$

leading to the optimal shortened dispersion relation

$$\left(\mathrm{h}_\mu + \frac{1}{3}\right)\mathrm{K}^2 - \frac{1}{\mu}\frac{\mathrm{h}_\mu}{\rho_\mu}\Omega^2 + \left(1 - \frac{\mu\rho_\mu}{3\mathrm{h}_\mu}\right) = 0.$$

Valid only over a narrow vicinity of the cut-off frequency!

Numerical illustration



$\mu = 0.014$, $\rho = 0.03$, and h = 1.0

Asymptotic formulae for displacements and stresses (setup A)

Leading order displacements and stresses

$$\begin{aligned} \mathrm{u_c} &= \mathrm{h_c}\xi_{2\mathrm{c}}, \\ \sigma_{13}^{\mathrm{c}} &= \mathrm{i}\mu_{\mathrm{c}}\sqrt{\mu}\,\mathrm{K}_*\xi_{2\mathrm{c}}, \\ \sigma_{23}^{\mathrm{c}} &= \mu_{\mathrm{c}}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{u_s} &= \mathrm{h_c}, \\ \sigma_{13}^{\mathrm{s}} &= \mathrm{i}\mu_{\mathrm{s}}\sqrt{\mu}\mathrm{K}_*, \\ \sigma_{23}^{\mathrm{s}} &= \mu_{\mathrm{c}}\mathrm{h}\left(\mathrm{K}_*^2 - \frac{\Omega_*^2}{\rho_\mu}\right)(\xi_{2\mathrm{s}} - 1)\,. \end{aligned}$$

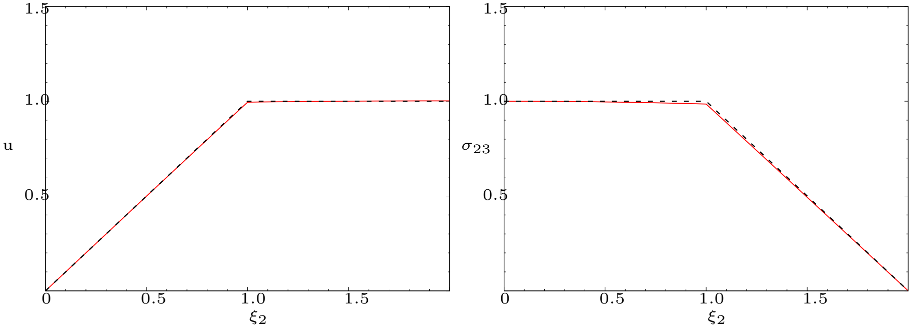
We obtain

$$\frac{\mathrm{u_q}}{\mathrm{h_c}} \sim \frac{\sigma_{23}^{\mathrm{q}}}{\mu_{\mathrm{c}}} \sim \frac{\sigma_{13}^{\mathrm{q}}}{\mu_{\mathrm{q}}\sqrt{\mu}}, \quad \mathrm{q} = \mathrm{c}, \mathrm{s}.$$

Normalised displacement and stress σ_{23} (setup A)

$$\xi_2 = \xi_{2c},\ \mathrm{u} = \frac{\mathrm{u_c}}{\mathrm{h_c}},\ \text{and } \sigma_{23} = \frac{\sigma_{23}^{\mathrm{c}}}{\mu_{\mathrm{c}}},\ (0 < \xi_2 \leq 1)$$

$$\text{or } \xi_2 = 1 + \xi_{2s},\ \mathrm{u} = \frac{\mathrm{u_s}}{\mathrm{h_c}},\ \text{and } \sigma_{23} = \frac{\sigma_{23}^{\mathrm{s}}}{\mu_{\mathrm{c}}},\ (1 < \xi_2 \leq 2)$$



Model construction (setup A)

Scaled longitudinal coordinate and time

$$\mathrm{x}_1 = \frac{\mathrm{h_c}}{\sqrt{\mu}}\xi_1 \quad \text{and} \quad \mathrm{t} = \frac{\mathrm{h_c}}{\mathrm{c_{2c}}\sqrt{\mu}}\tau,$$

Normalised displacement and stresses

$$\mathrm{u^q} = \mathrm{h_c v^q}, \quad \sigma_{13}^{\mathrm{q}} = \mu_{\mathrm{q}}\sqrt{\mu}\mathrm{S}_{13}^{\mathrm{q}}, \quad \sigma_{23}^{\mathrm{q}} = \mu_{\mathrm{c}}\mathrm{S}_{23}^{\mathrm{q}}, \quad \mathrm{q} = \mathrm{c}, \mathrm{s}.$$

with all dimensionless quantities assumed to be of order unity.

Core layer

$$\mu\frac{\partial \mathrm{S}_{13}^{\mathrm{c}}}{\partial \xi_1} + \frac{\partial \mathrm{S}_{23}^{\mathrm{c}}}{\partial \xi_{2\mathrm{c}}} - \mu\frac{\partial^2 \mathrm{v^c}}{\partial \tau^2} = 0,$$

$$\mathrm{S}_{13}^{\mathrm{c}} = \frac{\partial \mathrm{v^c}}{\partial \xi_1}, \quad \mathrm{S}_{23}^{\mathrm{c}} = \frac{\partial \mathrm{v^c}}{\partial \xi_{2\mathrm{c}}}.$$

Skin layer

$$\frac{\partial \mathrm{S}_{13}^{\mathrm{s}}}{\partial \xi_1} + \frac{1}{\mathrm{h}}\frac{\partial \mathrm{S}_{23}^{\mathrm{s}}}{\partial \xi_{2\mathrm{s}}} - \frac{1}{\rho_\mu}\frac{\partial^2 \mathrm{v^s}}{\partial \tau^2} = 0,$$

$$\mathrm{S}_{13}^{\mathrm{s}} = \frac{\partial \mathrm{v^s}}{\partial \xi_1}, \quad \mu \mathrm{h}\mathrm{S}_{23}^{\mathrm{s}} = \frac{\partial \mathrm{v^s}}{\partial \xi_{2\mathrm{s}}}.$$

Derivation of a shortened equation (setup A)

Continuity and boundary conditions

$$\mathrm{v}^{\mathrm{c}}\big|_{\xi_{2\mathrm{c}}=1} = \mathrm{v}^{\mathrm{s}}\big|_{\xi_{2\mathrm{s}}=0},$$
$$\mathrm{S}^{\mathrm{c}}_{23}\big|_{\xi_{2\mathrm{c}}=1} = \mathrm{S}^{\mathrm{s}}_{23}\big|_{\xi_{2\mathrm{s}}=0},$$

and

$$\mathrm{S}^{\mathrm{s}}_{23}\big|_{\xi_{2\mathrm{s}}=1} = 0.$$

Expand displacements and stresses into asymptotic series as

$$\mathrm{v}^{\mathrm{q}} = \mathrm{v}^{\mathrm{q}}_0 + \mu \mathrm{v}^{\mathrm{q}}_1 + \cdots,$$
$$\mathrm{S}^{\mathrm{q}}_{\mathrm{j}3} = \mathrm{S}^{\mathrm{q}}_{\mathrm{j}3,0} + \mu \mathrm{S}^{\mathrm{q}}_{\mathrm{j}3,1} + \cdots, \quad \mathrm{q} = \mathrm{c}, \mathrm{s} \quad \text{and} \quad \mathrm{j} = 1, 2.$$

Leading order problem

$$\mathrm{S}^{\mathrm{c}}_{13,0} = \frac{\partial \mathrm{v}^{\mathrm{c}}_0}{\partial \xi_1}, \quad \frac{\partial \mathrm{S}^{\mathrm{c}}_{23,0}}{\partial \xi_{2\mathrm{c}}} = 0, \quad \mathrm{S}^{\mathrm{c}}_{23,0} = \frac{\partial \mathrm{v}^{\mathrm{c}}_0}{\partial \xi_{2\mathrm{c}}},$$

and

$$\frac{\partial \mathrm{S}^{\mathrm{s}}_{13,0}}{\partial \xi_1} + \frac{1}{\mathrm{h}} \frac{\partial \mathrm{S}^{\mathrm{s}}_{23,0}}{\partial \xi_{2\mathrm{s}}} - \frac{1}{\rho_\mu} \frac{\partial^2 \mathrm{v}^{\mathrm{s}}_0}{\partial \tau^2} = 0,$$

$$\mathrm{S}^{\mathrm{s}}_{13,0} = \frac{\partial \mathrm{v}^{\mathrm{s}}_0}{\partial \xi_1}, \quad \frac{\partial \mathrm{v}^{\mathrm{s}}_0}{\partial \xi_{2\mathrm{s}}} = 0,$$

with

$$\mathrm{v}^{\mathrm{c}}_0\Big|_{\xi_{2\mathrm{c}}=1} = \mathrm{v}^{\mathrm{s}}_0\Big|_{\xi_{2\mathrm{s}}=0},$$

$$\mathrm{S}^{\mathrm{c}}_{23,0}\Big|_{\xi_{2\mathrm{c}}=1} = \mathrm{S}^{\mathrm{s}}_{23,0}\Big|_{\xi_{2\mathrm{s}}=0},$$

and

$$\mathrm{S}^{\mathrm{s}}_{23}\Big|_{\xi_{2\mathrm{s}}=1} = 0.$$

Leading order solution

$$v_0^s = w(\xi_1, \tau).$$

The rest of the quantities are expressed in terms of w as

$$S^c_{13,0} = \xi_{2c}\frac{\partial w}{\partial \xi_1}, \qquad S^c_{23,0} = w, \qquad v^c_0 = \xi_{2c} w,$$
$$S^s_{13,0} = \frac{\partial w}{\partial \xi_1}, \qquad S^s_{23,0} = w(1 - \xi_{2s}),$$

with w satisfying the 1D equation

$$\frac{\partial^2 w}{\partial \xi_1^2} - \frac{1}{\rho_\mu}\frac{\partial^2 w}{\partial \tau^2} - \frac{1}{h} w = 0,$$

which may be presented in the original variables as

$$\frac{\partial^2 u_s}{\partial x_1^2} - \frac{\rho_s}{\mu_s}\frac{\partial^2 u_s}{\partial t^2} - \frac{\mu_c}{\mu_s h_c h_s} u_s = 0,$$

where $u_s(x_1, t) \approx w(x_1, t)$.

Justification of the model

Insert ansatz $u_s = \mathsf{exp}\{i(kx_1 - \omega t)\}$ into the last equation. As a result, we have the dispersion relation

$$k^2 - \frac{\rho_s}{\mu_s}\omega^2 + \frac{\mu_c}{\mu_s h_c h_s} = 0.$$

Coincides with the shortened dispersion relation for setup A!

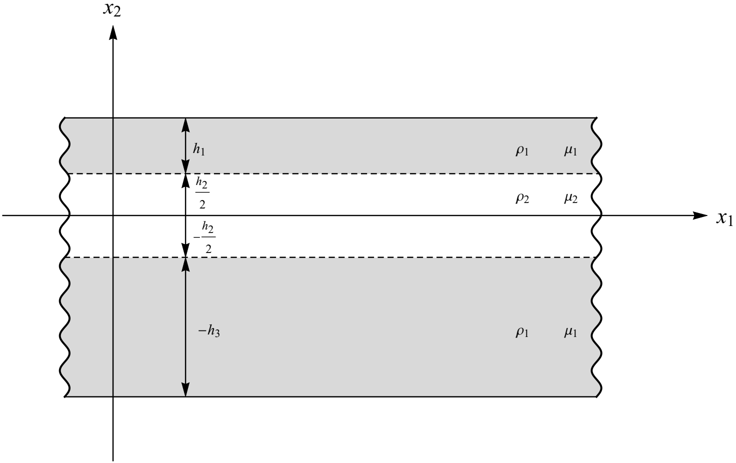
Contents

In-plane vector problem

Anti-plane scalar problem

Anti-plane scalar problem for asymmetric plates

Anti-plane shear of three-layered asymmetric plates



More sophisticated dispersion relation

$$\mu\alpha_1\alpha_2 \,\mathsf{tanh}(h\alpha_1) + \mu^2{\alpha_2}^2 \,\mathsf{tanh}(\alpha_2) + \\ \mu\alpha_1\alpha_2 \,\mathsf{tanh}(h^*\alpha_1) + {\alpha_1}^2 \,\mathsf{tanh}(h^*\alpha_1)\,\mathsf{tanh}(\alpha_2)\,\mathsf{tanh}(h\alpha_1) = 0,$$

where

$$\alpha_1 = \sqrt{K^2 - \frac{\mu}{\rho}\Omega^2}, \quad \alpha_2 = \sqrt{K^2 - \Omega^2},$$

with

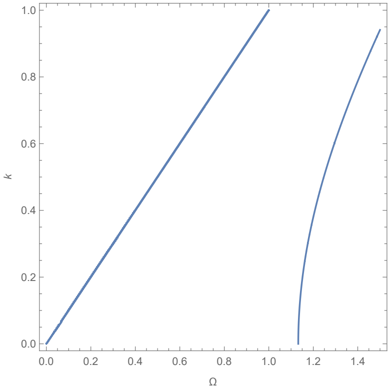
$$\Omega = \frac{\omega h_2}{c_2^{(2)}}, \quad K = kh_2,$$

and

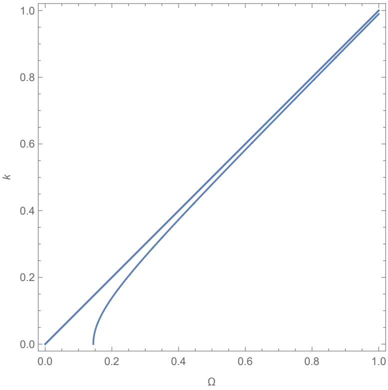
$$h = \frac{h_1}{h_2}, \quad h^* = \frac{h_3}{h_2}, \quad \mu = \frac{\mu_2}{\mu_1}, \quad \rho = \frac{\rho_2}{\rho_1}, \quad c_2^{(i)} = \sqrt{\frac{\mu_i}{\rho_i}}, \quad i = 1, 2$$

Effect of contrast

No contrast



Contrast parameters



Two modes in case of high contrast for a scalar problem!

Cut-off frequencies

Frequency equation

$$\sqrt{\mu\rho}\left(\tan\left(\mathrm{h}\sqrt{\frac{\mu}{\rho}}\Omega\right)+\tan\left(\mathrm{h}^*\sqrt{\frac{\mu}{\rho}}\Omega\right)\right)$$
$$+\,\mu\rho\tan(\Omega)-\tan\left(\mathrm{h}\sqrt{\frac{\mu}{\rho}}\Omega\right)\tan(\Omega)\tan\left(\mathrm{h}^*\sqrt{\frac{\mu}{\rho}}\Omega\right)=0.$$

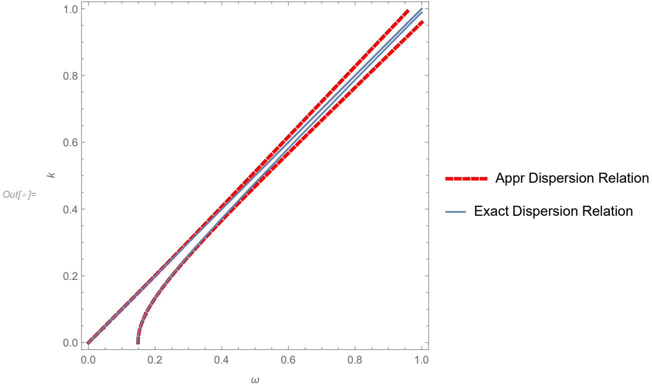
Lowest cut-off

$$\Omega\approx\sqrt{\frac{\mu\rho(\mathrm{h}+\mathrm{h}^*+\rho)}{\mathrm{h}\mathrm{h}^*\mu}}$$

A. Photovoltaic panels. Two-mode approximation

Shortened polynomial dispersion relation for two modes

$$G_1K^2+G_2\Omega^2+G_3K^4+G_4K^2\Omega^2+G_5\Omega^4+G_6K^4\Omega^2++G_7K^2\Omega^4=0$$



Concluding remarks

- Multi-parametric analysis is performed
- One- and two-mode approximations (both asymptotically uniform and composite) are constructed
- 1D shortened PDEs are derived for several setups.