The Directed Spanning Forest converges to the Brownian Web

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Coauthors

Figure: Kumarjit Saha, Anish Sarkar & Chi Tran.

Plan

1. The Directed Spanning Forest and its conjectures

2. The Brownian Web as a universal scaling limit

3. Some words about the proof

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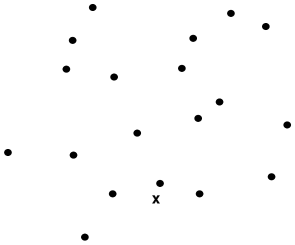
The Directed Spanning Forest $\mathcal{F}$ (DSF)

Vertex set: a homogeneous PPP $\mathcal{N}$ in $\mathbb{R}^2$.

$e_2 = (0, 1)$: a deterministic direction.

Local rule: each $\mathbf{x} \in \mathcal{N}$ is linked to the closest vertex in $\{z \in \mathbb{R}^2 : \langle z, X + e_2 \rangle \geq 0\}$.

$\Rightarrow$ The Directed Spanning Forest $\mathcal{F}$.



- Approximation (local and in distribution) of the *Radial Spanning Tree* studied by Baccelli & Bordenave ('08) to modelize communication networks.
- The DSF admits beautiful conjectures: Coalescence? Scaling limit ?
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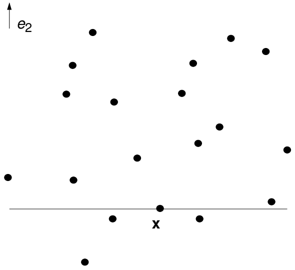

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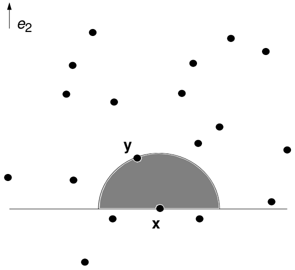

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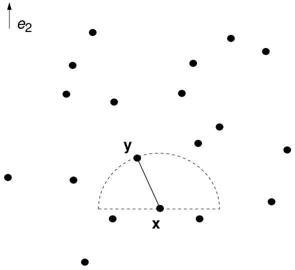

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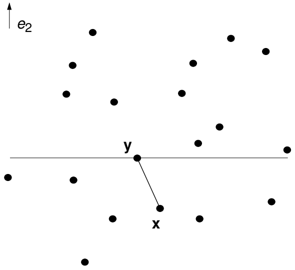

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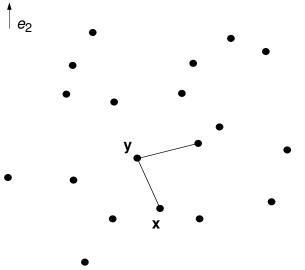

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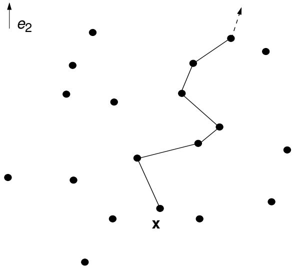

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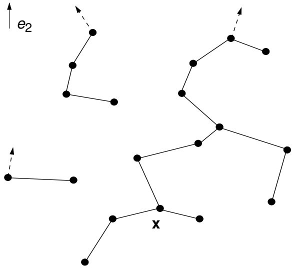

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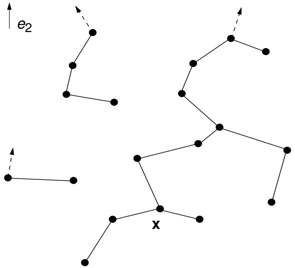

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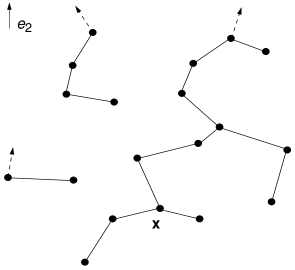

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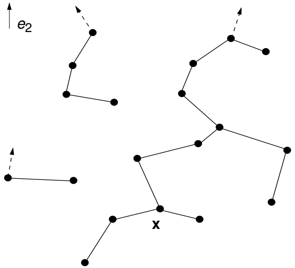

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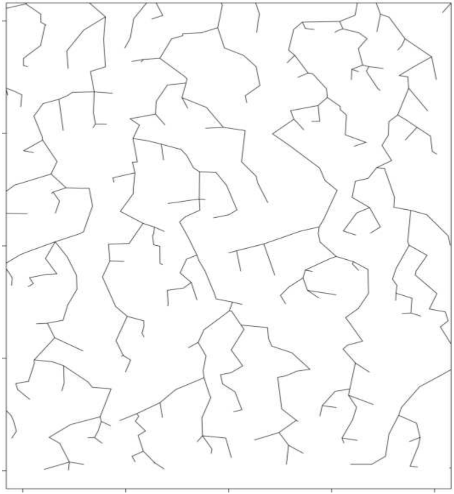
A simulation of the DSF



Dependence phenomenons

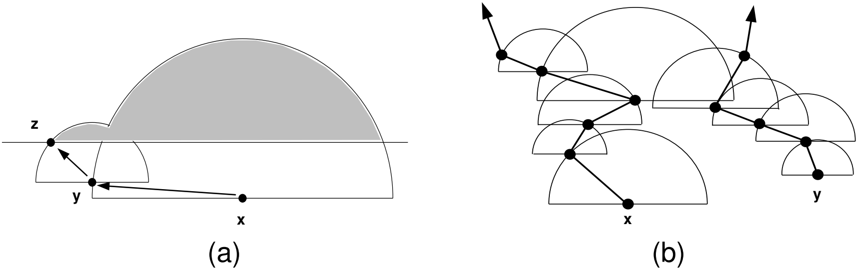


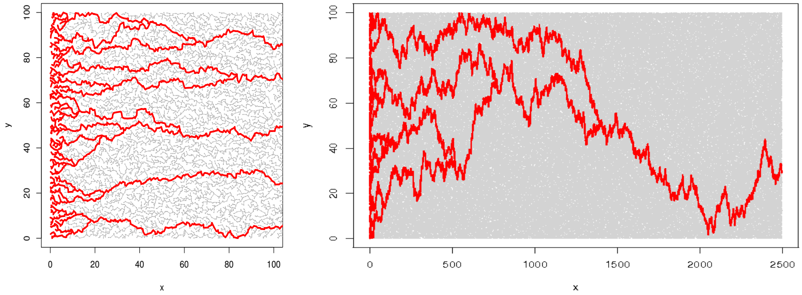
Figure: (a) Dependence phenomenon within a single path: how the past trajectory may influence its next steps. (b) Dependence phenomenon between two DSF trajectories: the overlap locally acts as a repulsive effect.

Coalescence

Theorem (C. & Tran '12)

(1) *A.s. all the DSF paths eventually coalesce.*

(2) *A.s. there is no bi-infinite path in the DSF $\mathcal{F}$.*



Scaling limit: our main result

- For $X := (X(1), X(2)) \in \mathcal{N}$, let $\pi^X : [X(2), \infty) \to \mathbb{R}$ be the linear interpolation of the DSF trajectory starting at X.
- Diffusive scaling: For $n \geq 1$, $\sigma, \gamma > 0$ and $X \in \mathcal{N}$, let

$$\pi_n^X(\sigma,\gamma)(\cdot) := \frac{1}{n\sigma}\pi^X(n^2\gamma\,\cdot)$$

and

$$\mathcal{X}_n(\sigma,\gamma) := \left\{\pi_n^X(\sigma,\gamma);\ X \in \mathcal{N}\right\}.$$

Theorem (C., Saha, Sarkar & Tran '18)

There exist $\sigma, \gamma > 0$ such that the sequence $\{\mathcal{X}_n(\sigma,\gamma), n \geq 1\}$ converges in distribution to the (standard) Brownian Web $\mathcal{W}$ as $n \to \infty$.

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Existence of the BW $\mathcal{W}$

Let $\Pi := \bigcup_{t_0 \in \mathbb{R}} C[t_0] \times \{t_0\}$ equipped with the distance:

$$d((f_1, t_1), (f_2, t_2)) := \left(\sup_t \left| \Phi(\widehat{f_1}(t), t) - \Phi(\widehat{f_2}(t), t) \right| \right) \vee |\Psi(t_1) - \Psi(t_2)|$$

$$\text{with } \Phi(x, t) := \frac{\tanh(x)}{1 + |t|} \text{ and } \Psi(t) := \tanh(t) .$$

Let $\mathcal{H}$ be the space of compact subsets of (Π, d), equipped with the Hausdorff metric.

Theorem (Fontes, Isopi, Newman & Ravishankar '04)

$\exists$ a $\mathcal{H}$-valued r.v. $\mathcal{W}$ whose distribution is uniquely determined by:

(i) for any $\mathbf{x} \in \mathbb{R}^2$, there is a.s. a unique path $\pi^{\mathbf{x}} \in \mathcal{W}$ starting from $\mathbf{x}$,

(ii) for any finite set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ of points, the collection $(\pi^{\mathbf{x}_1}, \ldots, \pi^{\mathbf{x}_k})$ is distributed as coalescing BMs starting from $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$,

(iii) for any countable deterministic dense subset $D \subset \mathbb{R}^2$, $\mathcal{W}$ a.s. is the closure of $\{\pi^{\mathbf{x}} : \mathbf{x} \in D\}$ in (Π, d).

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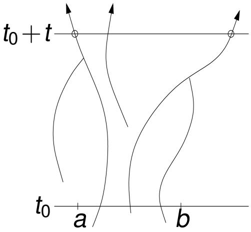
Convergence criteria

For $a, b, t_0 \in \mathbb{R}$, $t > 0$,

$$\eta_{\mathcal{X}}(t_0, t; a, b) := \#\Big\{\pi(t_0+t) : \pi \in \mathcal{X}, \pi(t_0) \in [a, b]\Big\}.$$



Theorem (Fontes, Isopi, Newman & Ravishankar '04)

A sequence $\{\mathcal{X}_n : n \geq 1\}$ of $\mathcal{H}-$valued r.v.'s with noncrossing paths converges to the BW if:

(I) For any countable dense set $D \subset \mathbb{R}^2$: for any $\mathbf{x} \in D$, there exists $\pi_n^{\mathbf{x}} \in \mathcal{X}_n$ s.t. for any finite subset $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset D$, $(\pi_n^{\mathbf{x}^1}, \ldots, \pi_n^{\mathbf{x}^k})$ converges in distribution to coalescing BMs started from $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

$(B1)$ $\forall t > 0$, $\overline{\lim}_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{\mathcal{X}_n}(t_0, t; a, a+\epsilon) \geq 2) \to 0$ as $\epsilon \downarrow 0$.

$(B2)$ $\forall t > 0$, $\frac{1}{\epsilon}\overline{\lim}_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{\mathcal{X}_n}(t_0, t; a, a+\epsilon) \geq 3) \to 0$ as $\epsilon \downarrow 0$.

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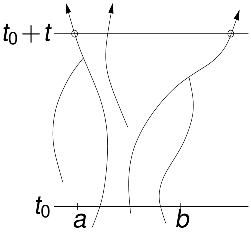

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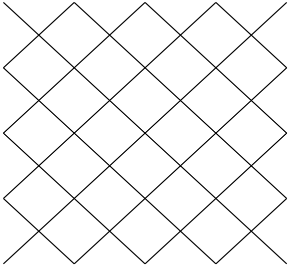
The coalescing random walks model

$$\mathbb{Z}^2_{\text{even}} := \Big\{(x,t) \in \mathbb{Z}^2; x+t \text{ is even}\Big\}.$$

Each vertex of $\mathbb{Z}^2_{\text{even}}$ goes to NE or NW, each with probability $\frac{1}{2}$ and independently.



Theorem (Fontes, Isopi, Newman & Ravishankar '04)

Under a diffusive scaling, this directed forest converges to the BW.

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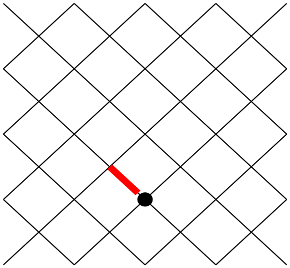

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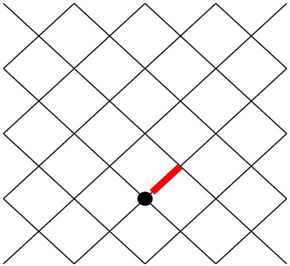

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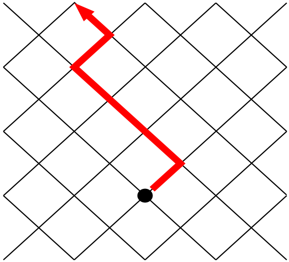

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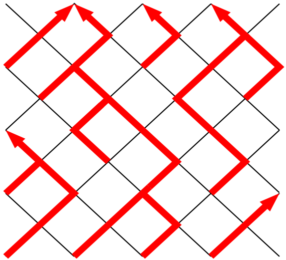

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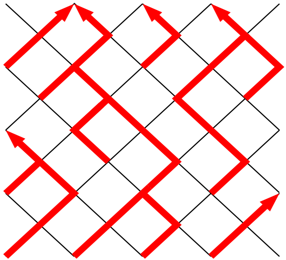

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The 2–dimensional Poisson Tree

Vertex set: a homogeneous PPP $\mathcal{N}$.

$e_2 = (0, 1)$: a deterministic direction.

$r > 0$: a deterministic parameter.

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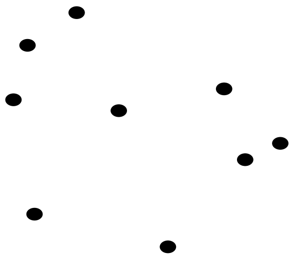

Theorem (Ferrari, Fontes & Wu '05)

Under a diffusive scaling, this directed forest converges to the BW.

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Vertex set: a homogeneous PPP $\mathcal{N}$.

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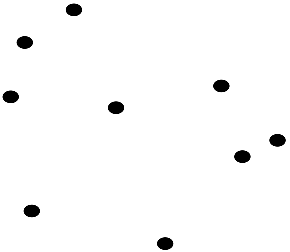

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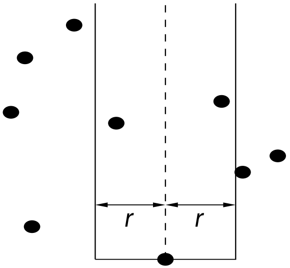

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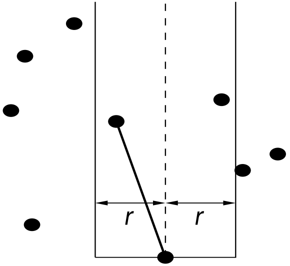

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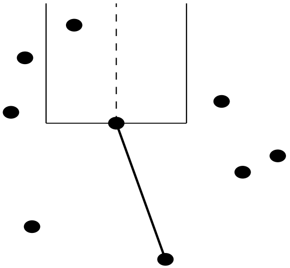

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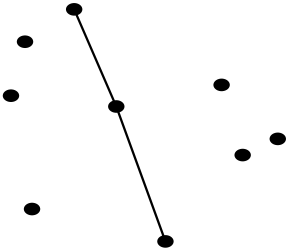

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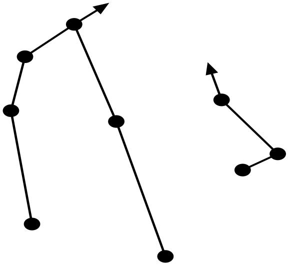

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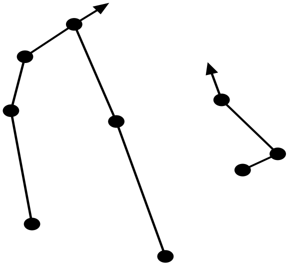

Theorem (Ferrari, Fontes & Wu '05)

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A discrete and L^1 DSF

V: site percolation on $\mathbb{Z}^2$ with parameter $0 < p < 1$.

Local rule: Each $\mathbf{u} \in V$ is linked to the closest $\mathbf{v} \in V$ in L^1–sense and with $\mathbf{v}(2) > \mathbf{u}(2)$.

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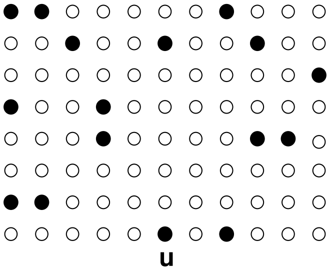

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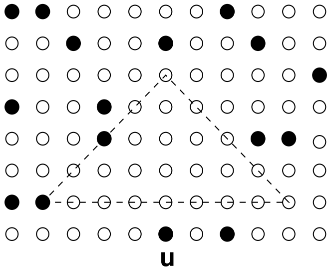

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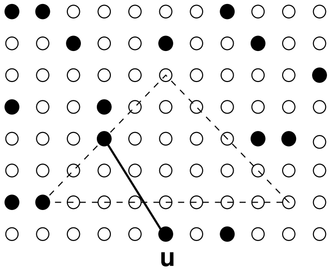

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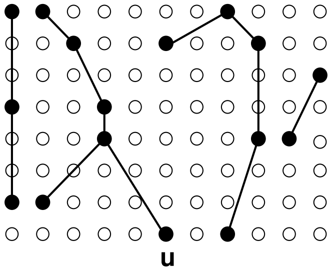

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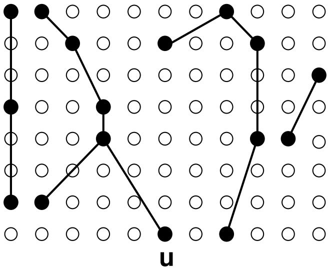

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Plan

1. The Directed Spanning Forest and its conjectures
2. The Brownian Web as a universal scaling limit
3. Some words about the proof

The 3 key points

- Schertzer et al '15: simplification of the convergence criteria for non-crossing paths.

  $(B2)$ and FKG inequality $\implies$ wedge condition.

- A coalescence time estimate based on a new Laplace type argument. If τ_ℓ is the coalescence time of DSF trajectories from $(0,0)$ and $(0,\ell)$,

$$\exists C > 0,\ \forall t \geq 0,\ \mathbb{P}(\tau_\ell > t) \leq \frac{C\ell}{\sqrt{t}} .$$

- Accurate study of the evolution of DSF paths: breaking points.

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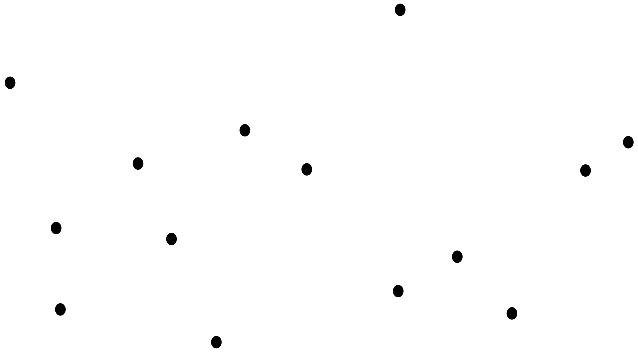
The joint exploration process



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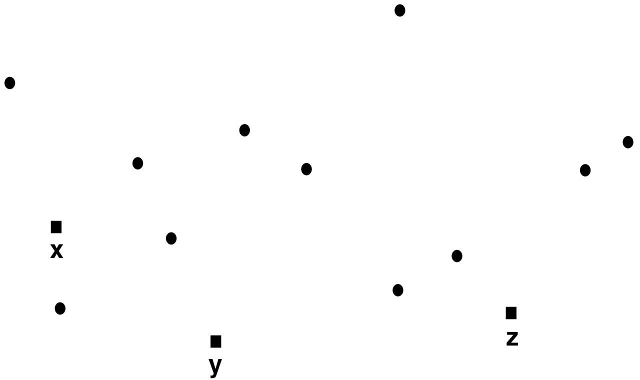

3 DSF paths starting from $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

$$(g_0(\mathbf{x}), g_0(\mathbf{y}), g_0(\mathbf{z})) = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \text{ and } H_0 = \emptyset.$$

H_n: History set and $L(H_n)$: height of H_n.

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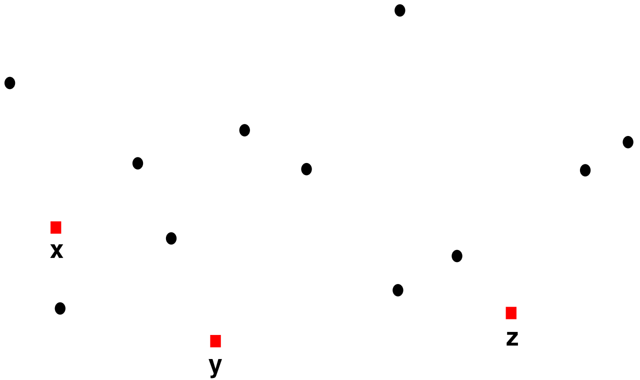

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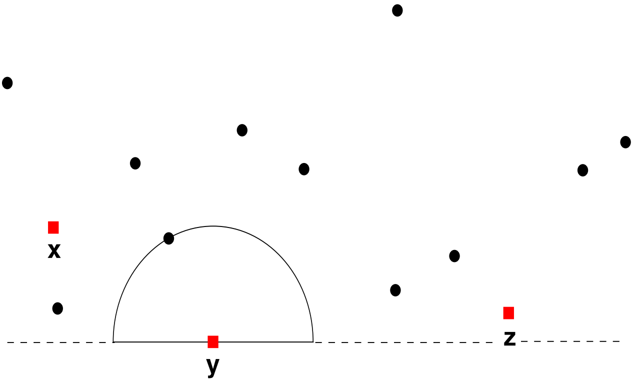

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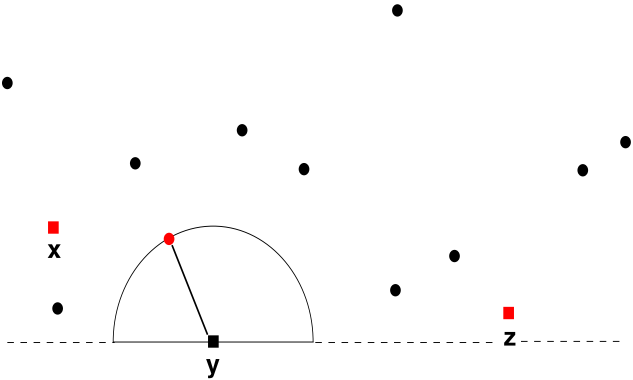

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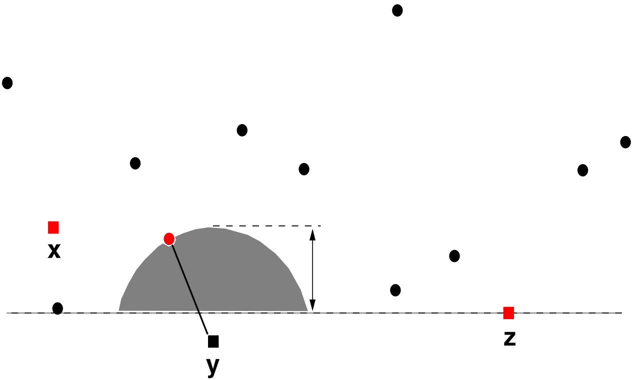

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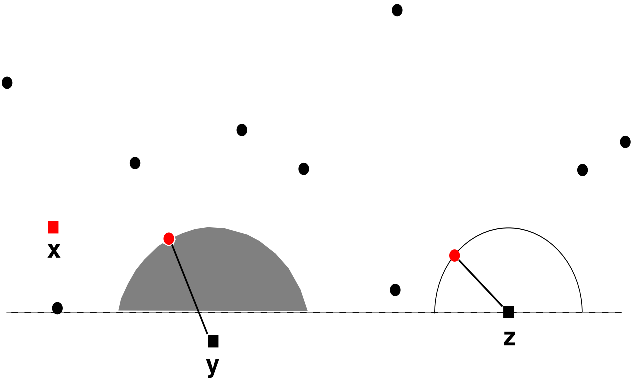

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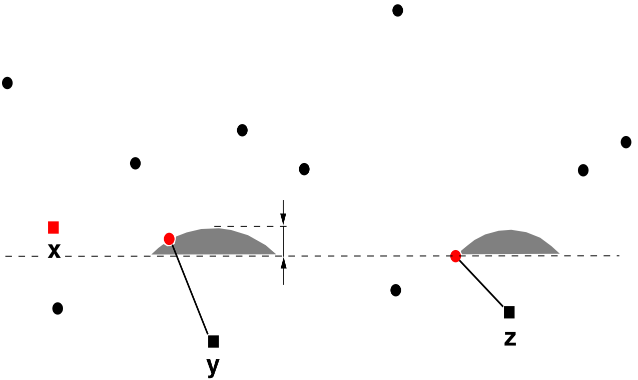

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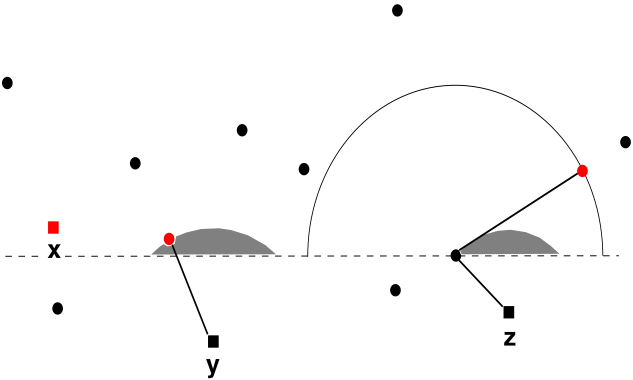

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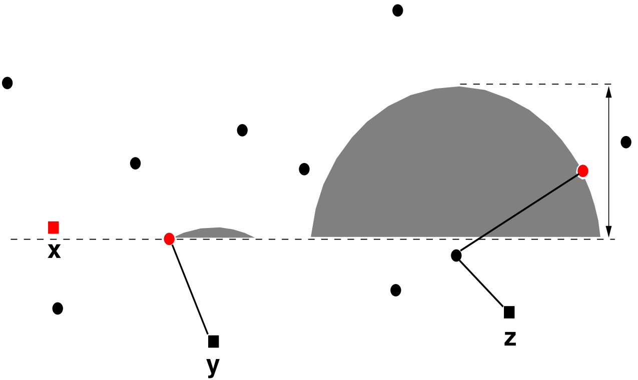

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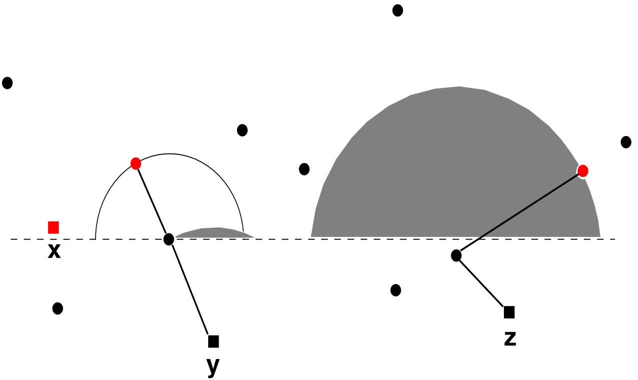

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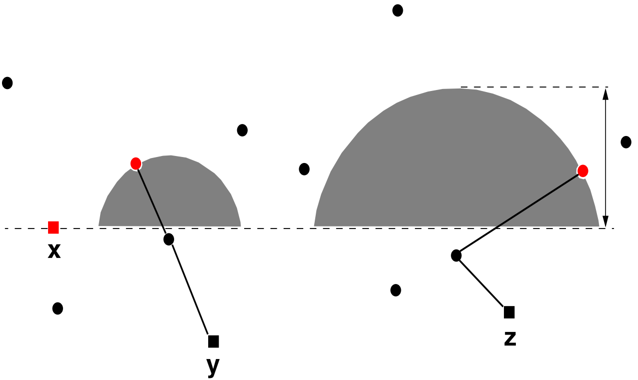

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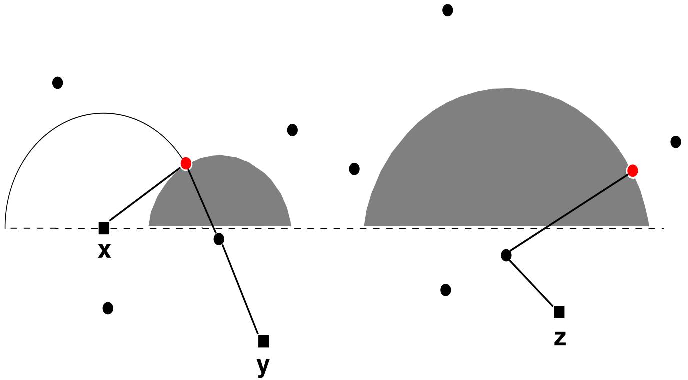

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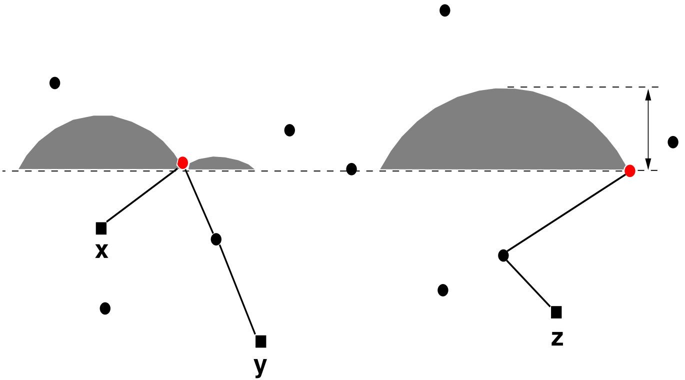

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Breaking points for k paths

Let $\kappa \geq 6$. The sequence of good steps $(\tau_j)_{j\geq 0}$ is defined by

$$\begin{cases} \tau_0 = 0, \\ \tau_{j+1} = \min\left\{kn > \tau_j : \ n \geq 1, \ L(H_{kn}) \leq \kappa \ \text{and} \ \dots \ \right\}, \ \text{for } j \geq 0. \end{cases}$$

Lemma

There exist $c, C > 0$ s.t. for any integers $j, n \geq 0$

$$\mathbb{P}\left(\tau_{j+1} - \tau_j \geq n \,|\, \mathcal{F}_{\tau_j}\right) \leq Ce^{-cn} ,$$

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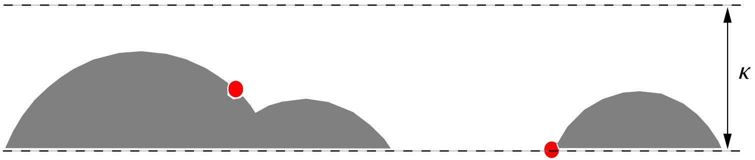


$(g_{\tau_j}(\mathbf{x}), g_{\tau_j}(\mathbf{y}))$ and H_{τ_j}

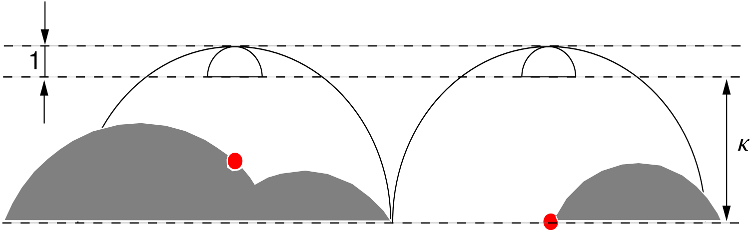
$(\mathbf{x}^{(\ell)}, \mathbf{y}^{(\ell)})$: restarting points of the ℓ-th perfect step.

Lemma

The sequence $\left\{(\mathbf{x}^{(\ell)}, \mathbf{y}^{(\ell)})\right\}_{\ell \geq 0}$ is a Markov chain.

Breaking points for $k = 2$ paths

The good step τ_j is a perfect step if



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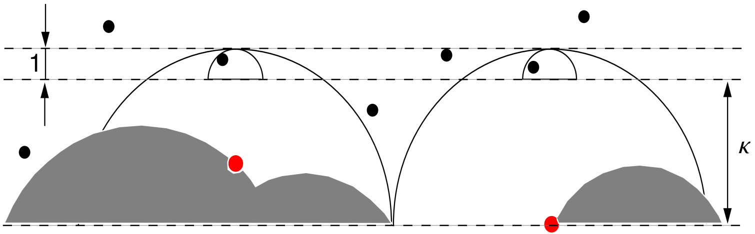

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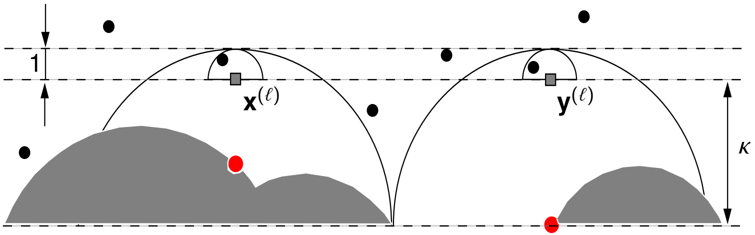

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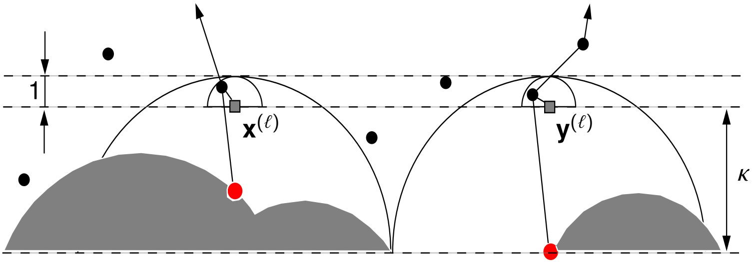

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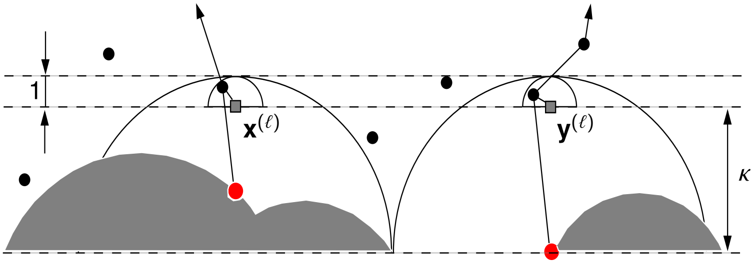

$$(g_{\tau_j}(\mathbf{x}), g_{\tau_j}(\mathbf{y})) \text{ and } H_{\tau_j}$$

$(\mathbf{x}^{(\ell)}, \mathbf{y}^{(\ell)})$: restarting points of the ℓ-th perfect step.

Lemma

The sequence $\left\{(\mathbf{x}^{(\ell)}, \mathbf{y}^{(\ell)})\right\}_{\ell \geq 0}$ is a Markov chain.