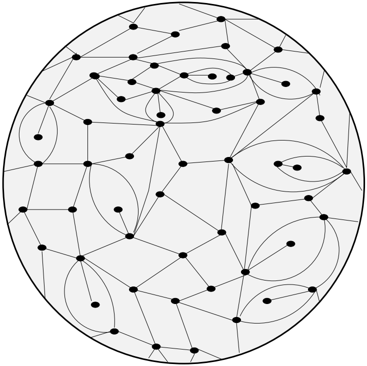
Some aspects of random maps coupled with matter systems

Jérémie Bouttier

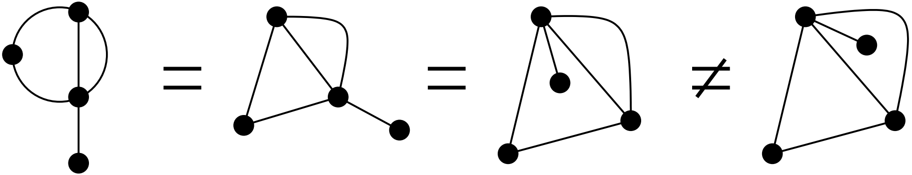
Institut de Physique Théorique, CEA Saclay
Laboratoire de Physique, ENS de Lyon

Workshop on Random Graphs and its Applications for Networks
Saint-Étienne, 3 October 2018

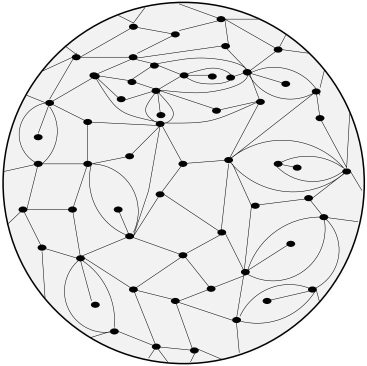
Maps: definition



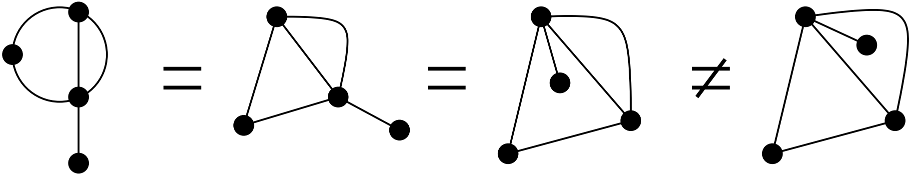
A planar map is a connected (multi)graph embedded in the sphere, considered up to continuous deformation. It is made of vertices, edges and faces.



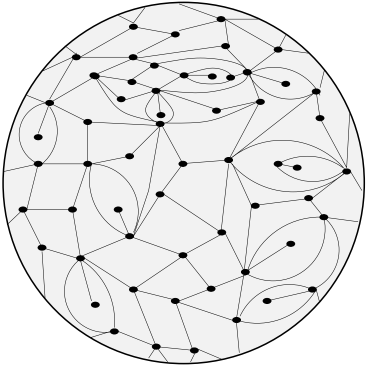
Maps: definition



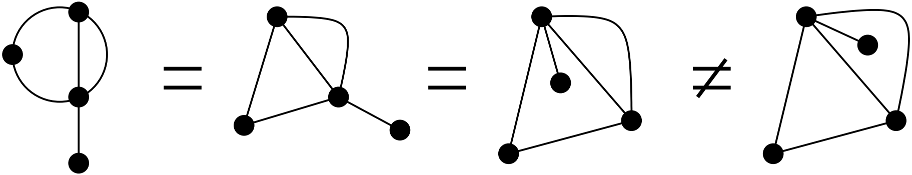
A planar map is a connected (multi)graph embedded in the sphere, considered up to continuous deformation. It is made of vertices, edges and faces.
When all faces have degree 4, the map is a quadrangulation. We similarly define triangulations, etc.



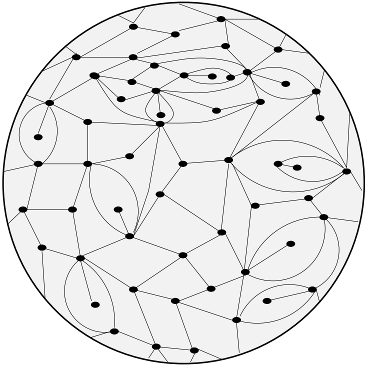
Maps: definition



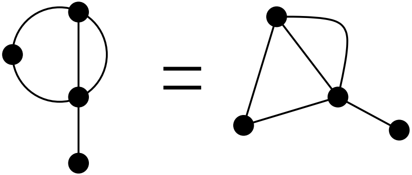
A planar map is a connected (multi)graph embedded in the sphere, considered up to continuous deformation. It is made of vertices, edges and faces.
When all faces have degree 4, the map is a quadrangulation. We similarly define triangulations, etc.
We may also consider maps on more general surfaces but we will not do so here.



Maps: definition



A planar map is a connected (multi)graph embedded in the sphere, considered up to continuous deformation. It is made of vertices, edges and faces.
When all faces have degree 4, the map is a quadrangulation. We similarly define triangulations, etc.
We may also consider maps on more general surfaces but we will not do so here.



Where maps arise

- Combinatorics (starting with W.T. Tutte)
  The number of rooted planar maps with n edges is

$$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{5/2}}.$$

  Also 4-color theorem, algorithmics (graph drawing)...

Where maps arise

- Combinatorics (starting with W.T. Tutte)
  The number of rooted planar maps with n edges is
  $$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{5/2}}.$$
  Also 4-color theorem, algorithmics (graph drawing)...
- Physics: large N expansion of gauge theories, matrix models, 2D quantum gravity... ($5/2$ is characteristic of “pure gravity”)

Where maps arise

- Combinatorics (starting with W.T. Tutte)
  The number of rooted planar maps with n edges is
  $$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{5/2}}.$$
  Also 4-color theorem, algorithmics (graph drawing)...
- Physics: large N expansion of gauge theories, matrix models, 2D quantum gravity... (5/2 is characteristic of "pure gravity")
- Algebraic geometry (ramified coverings, *dessins d'enfants*), representation theory (factorizations in symmetric group)...

Where maps arise

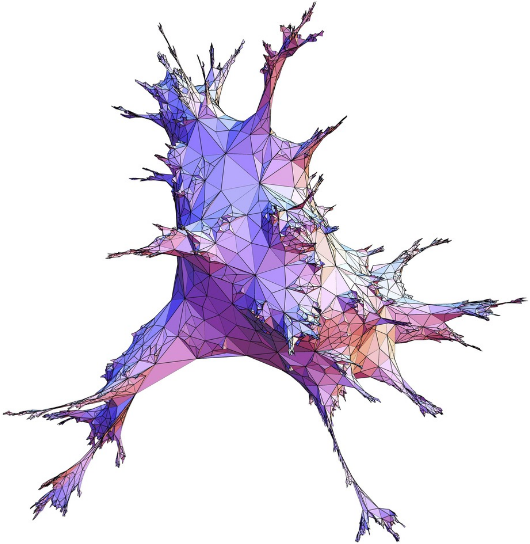
- Combinatorics (starting with W.T. Tutte)
  The number of rooted planar maps with n edges is

$$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{5/2}}.$$

  Also 4-color theorem, algorithmics (graph drawing)...

- Physics: large N expansion of gauge theories, matrix models, 2D quantum gravity... (5/2 is characteristic of "pure gravity")
- Algebraic geometry (ramified coverings, *dessins d'enfants*), representation theory (factorizations in symmetric group)...
- Probability theory: random geometry
  (5/2: maps whose scaling limit is the "Brownian map")

Random geometry



Picture of a large random triangulations by N. Curien

Towards the Brownian map

- Ambjørn-Watabiki (1995): predict that the "Hausdorff dimension" of pure gravity is 4, and compute the "two-point function".
- Chassaing-Schaeffer (2002): the radius of a random rooted quadrangulation with n faces, rescaled by $n^{-1/4}$, converges in law.
- B., Guitter, Di Francesco (2002-2010): similar results for other observables or families of maps: two-point and three-point functions, number of geodesics, loops, maps with boundaries...
- Marckert-Mokkadem (2006): definition of the Brownian map, proof of convergence of rescaled quadrangulations in a weak sense.
- Le Gall, Miermont and others (2007-2013): proof of the convergence of rescaled p-angulations ($p = 3, 4, 6, \ldots$) to the Brownian map in the Gromov-Hausdorff sense.
- Miller-Sheffield (2013-2016): connection with Liouville quantum gravity at $\gamma = \sqrt{8/3}$ via Quantum Loewner evolution.

What else?

It is widely believed that other scaling limits are obtained for maps decorated with a critical statistical physics system ("matter").

What else?

It is widely believed that other scaling limits are obtained for maps decorated with a critical statistical physics system ("matter").

The first evidence is Kazakov's exact solution of the Ising model on triangulations or quadrangulation, for which a different counting exponent ($7/3$ rather than $5/2$) is obtained.

What else?

It is widely believed that other scaling limits are obtained for maps decorated with a critical statistical physics system ("matter").

The first evidence is Kazakov's exact solution of the Ising model on triangulations or quadrangulation, for which a different counting exponent ($7/3$ rather than $5/2$) is obtained.

But the geometry is much less understood, for instance it is not known what the Hausdorff dimension should be: 4? 4.212 (Watabiki)?
Ding-Gwynne bounds (2018): between 4.189 and 4.243?

What else?

It is widely believed that other scaling limits are obtained for maps decorated with a critical statistical physics system ("matter").

The first evidence is Kazakov's exact solution of the Ising model on triangulations or quadrangulation, for which a different counting exponent ($7/3$ rather than $5/2$) is obtained.

But the geometry is much less understood, for instance it is not known what the Hausdorff dimension should be: 4? 4.212 (Watabiki)?
Ding-Gwynne bounds (2018): between 4.189 and 4.243?

There should be a one-parameter family of scaling limits: γ-LQG surfaces with $\gamma \in (0,2)$ (pure gravity: $\gamma = \sqrt{8/3}$, Ising: $\gamma = \sqrt{3}$...).

What else?

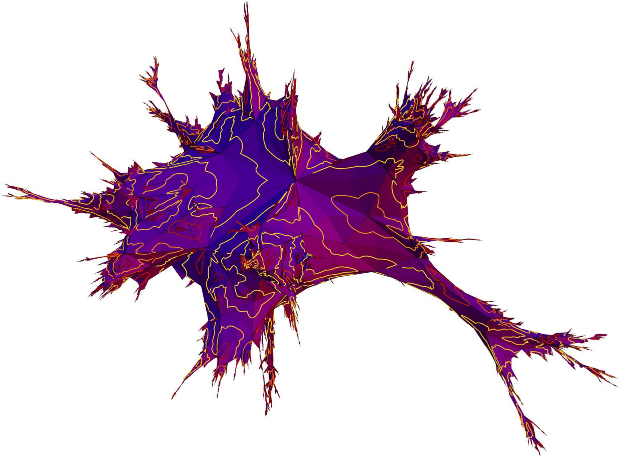
It is widely believed that other scaling limits are obtained for maps decorated with a critical statistical physics system ("matter").

The first evidence is Kazakov's exact solution of the Ising model on triangulations or quadrangulation, for which a different counting exponent ($7/3$ rather than $5/2$) is obtained.

But the geometry is much less understood, for instance it is not known what the Hausdorff dimension should be: 4? 4.212 (Watabiki)? Ding-Gwynne bounds (2018): between 4.189 and 4.243?

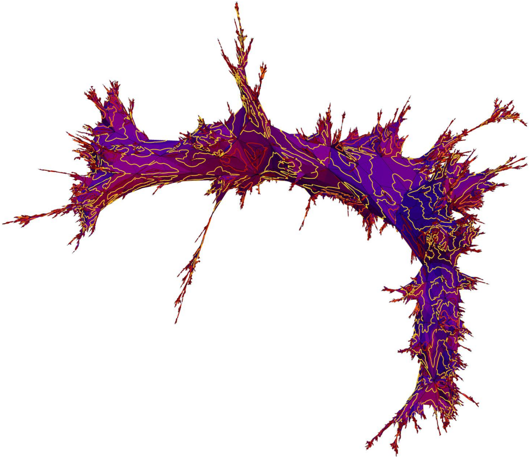
There should be a one-parameter family of scaling limits: γ-LQG surfaces with $\gamma \in (0, 2)$ (pure gravity: $\gamma = \sqrt{8/3}$, Ising: $\gamma = \sqrt{3}$...). It should be obtained as limits of maps decorated with a critical q-state Potts model or q-FK percolation, $q \in (0, 4)$, or with a $O(n)$ loop model, $n \in (0, 2)$.

What else?



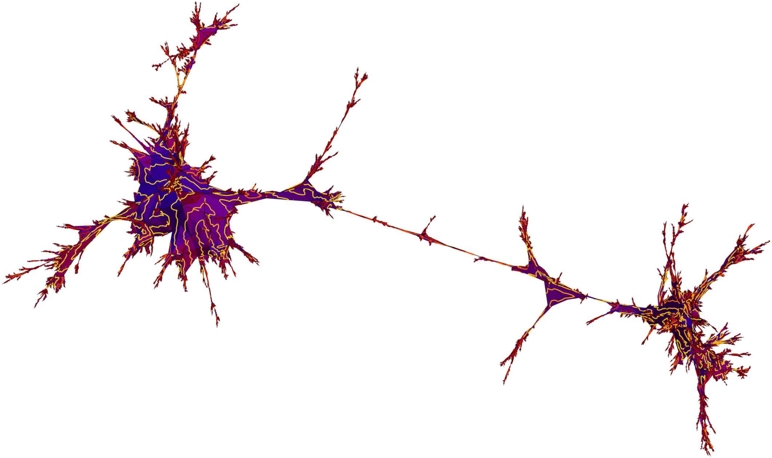
Picture of a critical FK-weighted triangulation by J. Bettinelli

What else?



Picture of a critical FK-weighted triangulation by J. Bettinelli

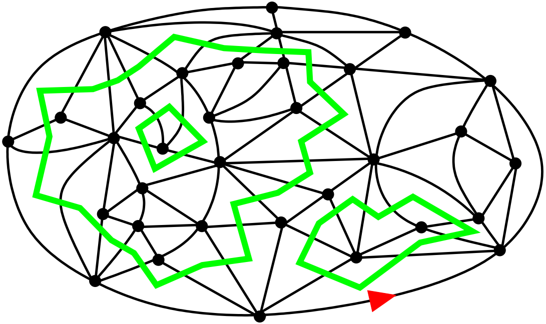
What else?



Picture of a critical FK-weighted triangulation by J. Bettinelli

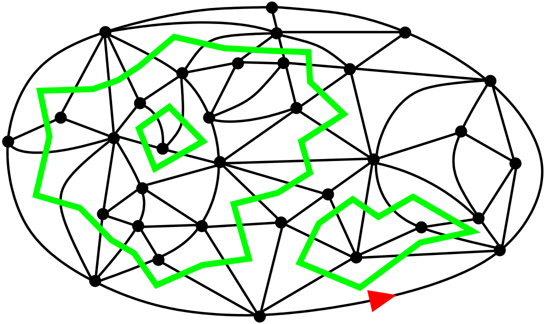
The $O(n)$ loop model on random maps

Given a map, a loop configuration is a collection of disjoint simple cycles (loops) on the dual map.



The $O(n)$ loop model on random maps

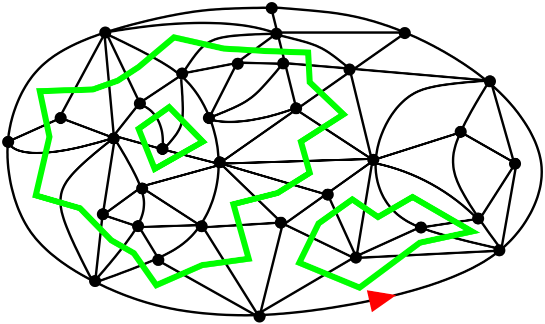
Given a map, a loop configuration is a collection of disjoint simple cycles (loops) on the dual map.



A configuration of the $O(n)$ loop model is a map endowed with a loop configuration. We assign it a weight $n^{\#\text{loops}} \times$ (face weights).

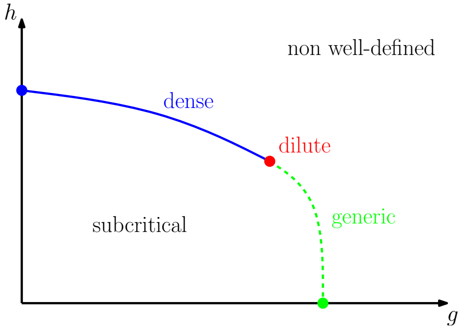
The $O(n)$ loop model on random maps

Given a map, a loop configuration is a collection of disjoint simple cycles (loops) on the dual map.



A configuration of the $O(n)$ loop model is a map endowed with a loop configuration. We assign it a weight $n^{\#\text{loops}} \times$ (face weights).
For instance for triangulations the weight is $n^{\#\text{loops}} g^{\#\triangle} h^{\#\triangle}$.

Phase diagram for $n \in (0, 2)$



We find three types of critical points:

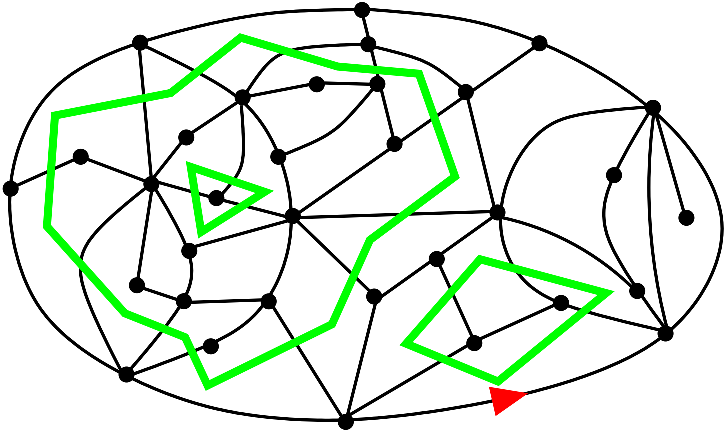
- generic (pure gravity, counting exponent $5/2$),
- dilute (counting exponent $2 + b$),
- dense (counting exponent $2 + b/(1 - b)$),

where $b := \frac{1}{\pi} \arccos\left(\frac{n}{2}\right) \in (0, 1/2)$.

Some works on the $O(n)$ loop model on random maps

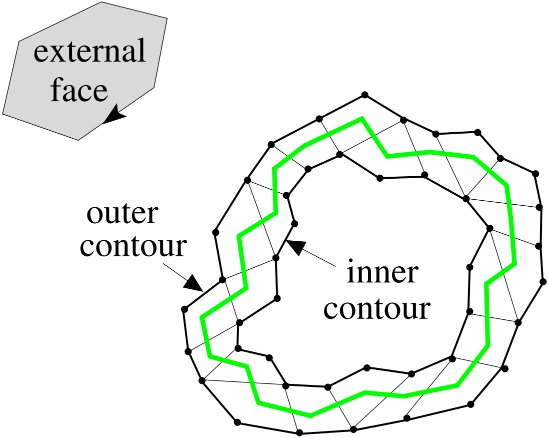
- 1990's: studied in physics via matrix models (Kostov, Eynard...)
- Le Gall-Miermont (2009): conjectured a connection with "stable maps"
- Borot-B.-Guitter (2011-2012): proved the connection using the "gasket decomposition"
- Borot-B.-Duplantier and Curien-Chen-Maillard (2016): study of nesting statistics and the perimeter cascade
- Budd (2018): peeling process and application to perimeters and FPP-type distance.

The gasket decomposition



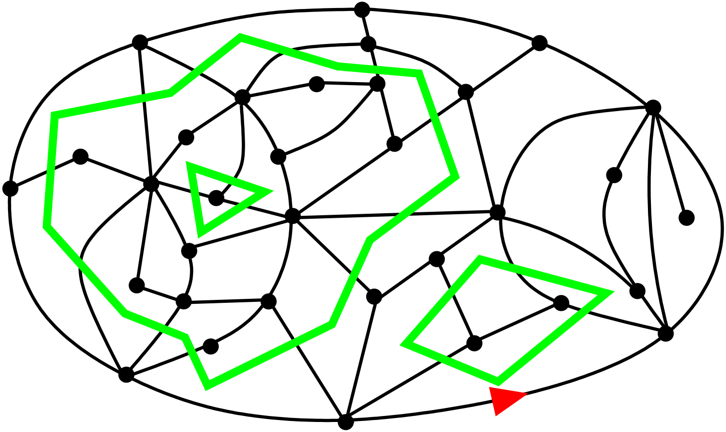
Start with a configuration of the O(n) loop model.

The gasket decomposition



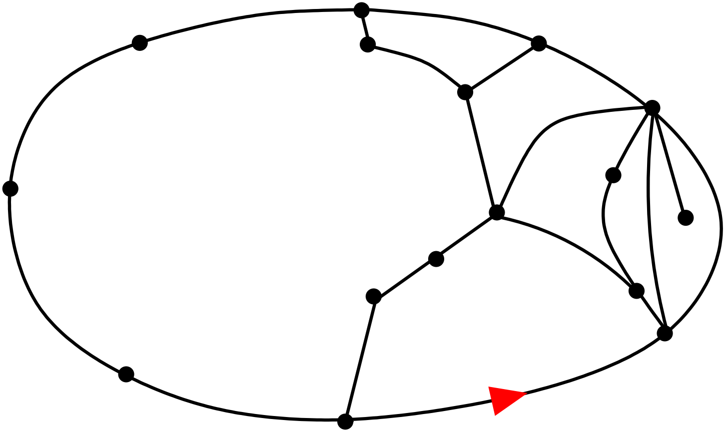
The faces visited by a loop forms a necklace.

The gasket decomposition



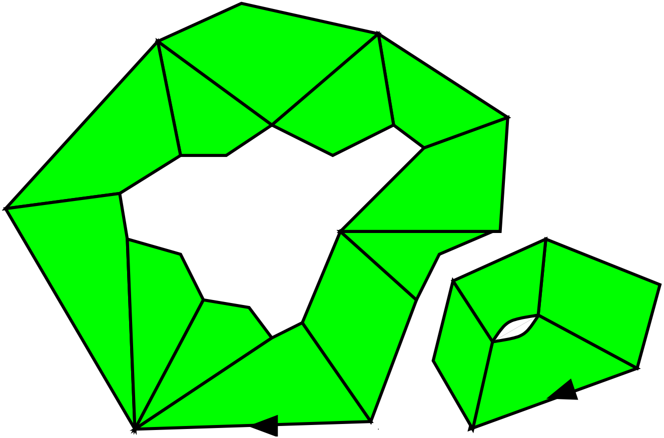
Cut along the outer and inner contours of each outermost loop.

The gasket decomposition



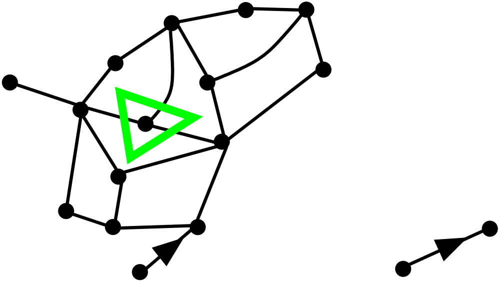
The outer component forms the gasket. It is a map without loops, with the same outer degree as the original map.

The gasket decomposition



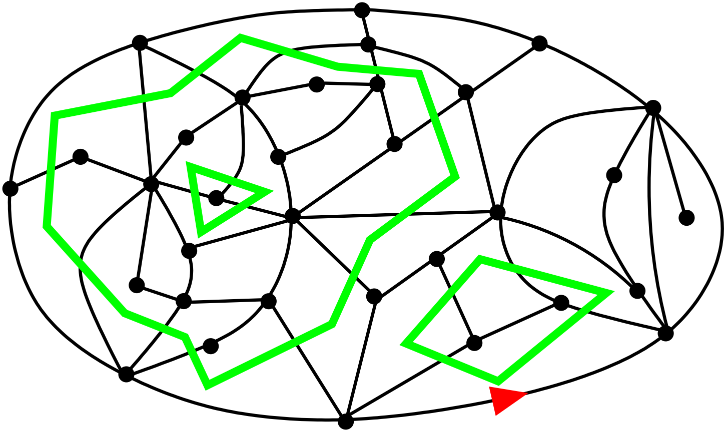
Each outermost loop forms a necklace (cyclic sequence of polygons glued side-by-side).

The gasket decomposition



Each outermost loop contains an internal configuration (of the same nature as our original object).

The gasket decomposition



The decomposition is bijective: given the gasket, necklaces and internal configurations, we may reconstruct the initial configuration.

The gasket decomposition: consequences

A corollary is that the gasket is a Boltzmann random map: the probability to observe a given map m is proportional to

$$w(m) = \prod_{f \text{ face of } m} g_{\text{degree}(f)}$$

where $(g_k)_{k \geq 1}$ is a sequence of parameters solution of a consistency (fixed point) equation, fixing it in terms of the parameters n, g, h of the model.

The gasket decomposition: consequences

A corollary is that the gasket is a Boltzmann random map: the probability to observe a given map m is proportional to

$$w(m) = \prod_{f \text{ face of } m} g_{\text{degree}(f)}$$

where $(g_k)_{k\geq 1}$ is a sequence of parameters solution of a consistency (fixed point) equation, fixing it in terms of the parameters n, g, h of the model.

Boltzmann random maps were investigated by Le Gall and Miermont (and more recently by Marzouk), who showed that for some fine-tuned sequences $(g_k)_{k\geq 1}$ we obtain a nongeneric scaling limit different from the Brownian map.

The gasket decomposition: consequences

A corollary is that the gasket is a Boltzmann random map: the probability to observe a given map m is proportional to

$$w(m) = \prod_{f \text{ face of } m} g_{\text{degree}(f)}$$

where $(g_k)_{k\geq 1}$ is a sequence of parameters solution of a consistency (fixed point) equation, fixing it in terms of the parameters n, g, h of the model.

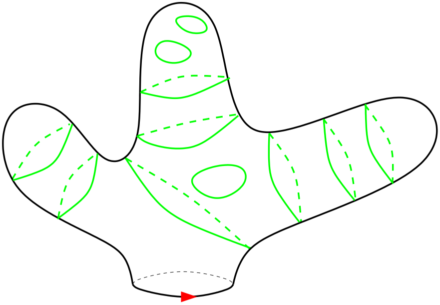
Boltzmann random maps were investigated by Le Gall and Miermont (and more recently by Marzouk), who showed that for some fine-tuned sequences $(g_k)_{k\geq 1}$ we obtain a nongeneric scaling limit different from the Brownian map.

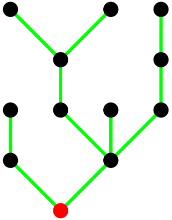
With Borot and Guitter we showed that such nongeneric scaling limits are precisely obtained at the nongeneric critical points of the $O(n)$ loop model, and we deduced the gasket dimension

$$d_H^{\text{gasket}} = \begin{cases} 3+2b & \text{(dilute)}, \\ 3-2b & \text{(dense)}. \end{cases}$$

Nesting tree

Rather than considering distances (which are hard to understand), we may study the structure of nestings between loops. In the planar case they are coded by the nesting tree.

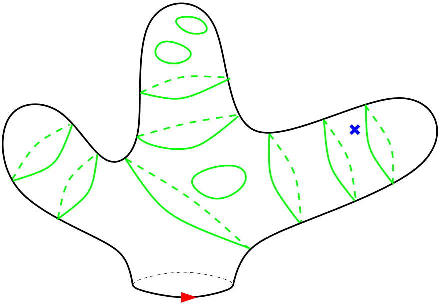


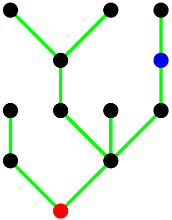


Each node of the nesting tree corresponds to a map without loops but arbitrarily large faces. In particular the root of the tree corresponds to the gasket. What is the structure of the nesting tree at a critical point?

Nesting tree

A simpler question: given an $O(n)$ configuration on a map with a boundary, what is the distribution of the depth of a uniformly chosen vertex ? (i.e. the number of loops separating it from the outer face, or the height of the corresponding node in the nesting tree)





Nesting statistics

At a non generic critical point, the depth grows logarithmically with the "size", and more precisely:

Theorem 1 (central limit theorem) [Borot-B.-Duplantier 2016]

Let P_ℓ be the depth of a uniformly chosen vertex in a random configuration of perimeter ℓ. Then, at a non generic critical point, we have

$$\frac{P_\ell - \frac{p_{\rm opt}}{\pi} \ln \ell}{\sqrt{\ln \ell}} \xrightarrow[\ell \to \infty]{(d)} \mathcal{N}(0, \sigma^2)$$

where

$$p_{\rm opt} = \frac{n}{\sqrt{4 - n^2}}, \qquad \sigma^2 = \frac{4n}{\pi(4 - n^2)^{3/2}}.$$

Nesting statistics

Theorem 2 (large deviation principle) [Borot-B.-Duplantier 2016]

Let P_ℓ be the depth of a uniformly chosen vertex in a random configuration of perimeter ℓ. Then, at a non generic critical point, we have

$$\mathbb{P}\left(P_\ell = \frac{\ln \ell}{\pi} p\right) \sim \mathrm{cst} \cdot (\ln \ell)^{-1/2} \ell^{-\frac{1}{\pi} J(p)}, \qquad \ell \to \infty$$

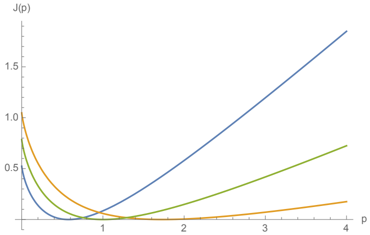
where

$$J(p) = p \ln \left(\frac{2}{n} \frac{p}{\sqrt{1+p^2}} \right) + \operatorname{arccot}(p) - \arccos(n/2).$$

Nesting statistics

Remarks:

- We have similar statements when the perimeter ℓ is replaced by the volume (number of vertices).
- Instead of marking a vertex, we may mark an inner face and obtain similar results.
- The large deviation function $J(p)$ is nonnegative, vanishes at $p = p_{\mathrm{opt}}$, satisfies $J''(p) = \frac{1}{p(p^2+1)}$, $J(p) \sim p \ln(2/n)$ for $p \to \infty$ and $J(0) = \arcsin(n/2) = \pi(1/2 - b)$ (consistently with the value of ν given before).



Nesting statistics

Our results can be compared to the so-called *multifractal spectrum of extreme nesting of* CLE_κ obtained by Miller, Watson and Wilson (2016).

Nesting statistics

Our results can be compared to the so-called *multifractal spectrum of extreme nesting of CLE*$_\kappa$ obtained by Miller, Watson and Wilson (2016). Colloquially speaking it amounts to a large deviation principle for the number of loops surrounding a small ball of Euclidean radius ϵ, as $\epsilon \to 0$. Our comparison then consists in replacing the constraint on the Euclidean radius by one on the *Liouville quantum area* δ.

Theorem 3 [MWW+BBD'16]

Let $\mathcal{N}_\delta$ be the number of loops surrounding a small ball of quantum area δ in a CLE$_\kappa$ coupled to Liouville quantum gravity (for suitable γ) on the Riemann sphere. Then we have

$$\mathbb{P}\left(\mathcal{N}_\delta = \frac{cp}{\pi}\ln(1/\delta)\right) \sim C\delta^{\frac{c}{\pi}J(p)}, \qquad \delta \to 0$$

with c, $J(p)$ as in Theorem 2.

Nesting statistics

Our results can be compared to the so-called *multifractal spectrum of extreme nesting of CLE*$_\kappa$ obtained by Miller, Watson and Wilson (2016). Colloquially speaking it amounts to a large deviation principle for the number of loops surrounding a small ball of Euclidean radius ϵ, as $\epsilon \to 0$. Our comparison then consists in replacing the constraint on the Euclidean radius by one on the *Liouville quantum area* δ.

Theorem 3 [MWW+BBD'16]

Let $\mathcal{N}_\delta$ be the number of loops surrounding a small ball of quantum area δ in a CLE$_\kappa$ coupled to Liouville quantum gravity (for suitable γ) on the Riemann sphere. Then we have

$$\mathbb{P}\left(\mathcal{N}_\delta = \frac{cp}{\pi}\ln(1/\delta)\right) \sim C\delta^{\frac{c}{\pi}J(p)}, \qquad \delta \to 0$$

with c, $J(p)$ as in Theorem 2.

This supports the conjecture that the scaling limit of the critical $O(n)$ model on random maps is described by a CLE coupled to LQG.