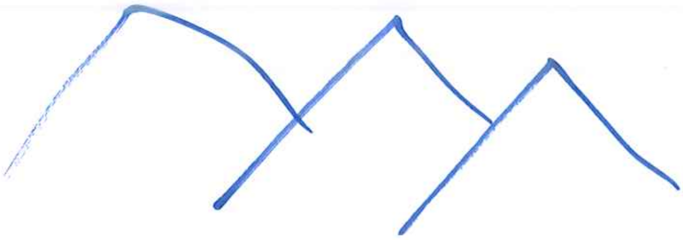

St Etienne
2018

RANDOM
GRAPHS
COMPLEX
NETWORKS

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Equations on a graph for the flows in thin tube structures

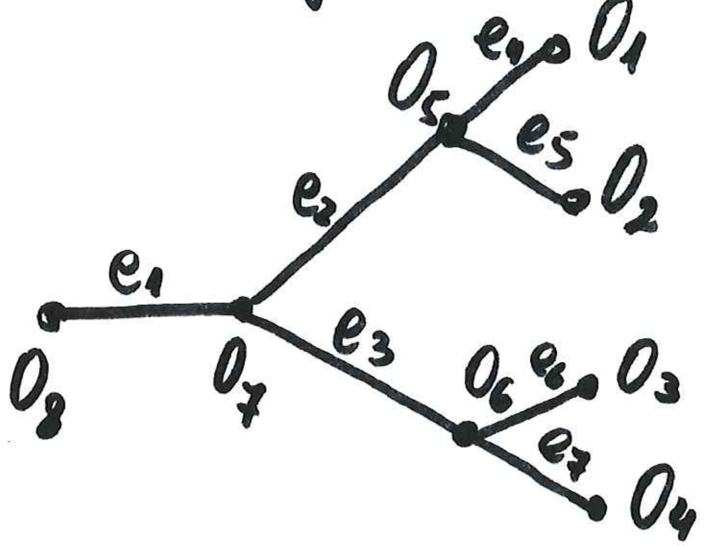
THIN STRUCTURE

1. Graph. $O_1, \dots, O_N \in \mathbb{R}^n$, $n \in \{2, 3\}$

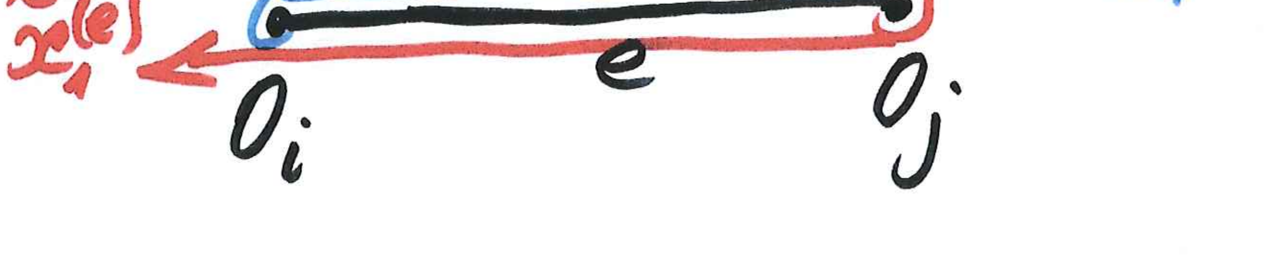
$e_1, \dots, e_M$ closed segments

$e_j = \overline{O_{i_j} O_{k_j}}$, $e_{j_1} \cap e_{j_2} \subseteq \{O_1, \dots, O_N\}$

$$\mathcal{B} = \bigcup_{j=1}^{M} e_j$$



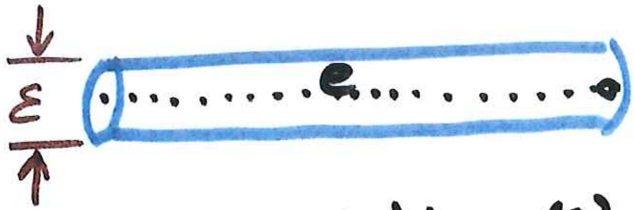
2. Local coordinate systems



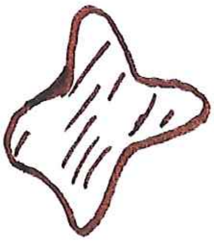
3. Thin structure

$e_j = e$

 $\sigma^j = \sigma^{(e)}$



$$B_\varepsilon^{(e)} = \Big\{ x^{(e)} \,\Big|\, x_1^{(e)} \in (0, |e|),\ \underbrace{\frac{x^{(e)\prime}}{\varepsilon} \in \sigma^{(e)}}_{\boxed{x^{(e)\prime} \in \sigma_\varepsilon^{(e)}}} \Big\}$$

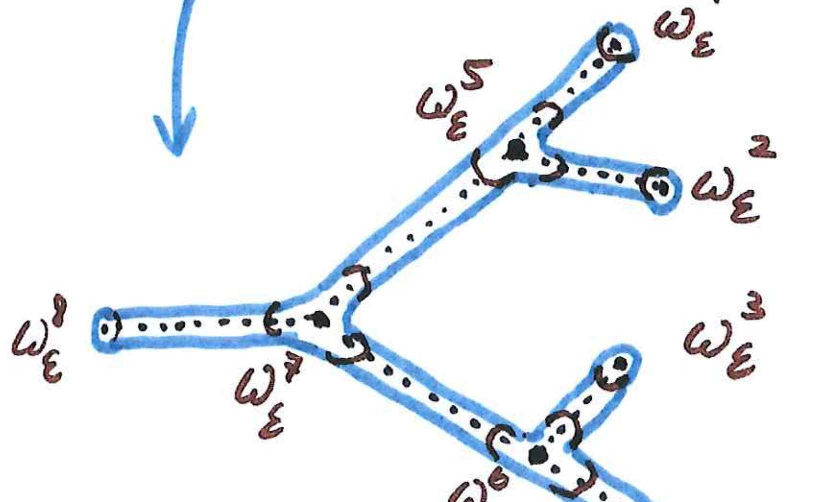
 ω^i bounded domains in $\mathbb{R}^n$, $i = 1, \dots, N$

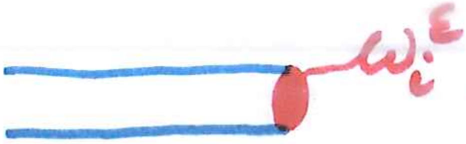
 $\omega_\varepsilon^i = \Big\{ x \in \mathbb{R}^n \,\Big|\, \frac{x - O_i}{\varepsilon} \in \omega^i \Big\}$

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} B_\varepsilon^{(e_j)} \Big) \cup \Big(\bigcup_{j=1}^{N} \omega_\varepsilon^j \Big)$$

connected

$\partial B_\varepsilon \in C^2$





$\gamma_\varepsilon^i = \partial\omega_\varepsilon^i \cap \partial B_\varepsilon$

$i = N_1+1, \dots, N$: O_i vertices

4. Navier - Stokes equations

$$
\left\{
\begin{aligned}
&\frac{1}{\varepsilon^2}\frac{\partial \vec{u}_\varepsilon}{\partial t} - \nu \Delta \vec{u}_\varepsilon + (\vec{u}_\varepsilon, \nabla)\vec{u}_\varepsilon + \nabla p = 0 \\
&\qquad\qquad\qquad x \in B_\varepsilon,\ t \in \mathbb{R} \\
&\operatorname{div} \vec{u}_\varepsilon = 0 \\
&\vec{u}_\varepsilon = \vec{g}_i\left(\frac{x - O_i}{\varepsilon}, t\right),\ x \in \gamma_\varepsilon^i \\
&\vec{u}_\varepsilon = 0,\ x \in \partial B_\varepsilon \setminus \gamma_\varepsilon,\ \gamma_\varepsilon = \bigcup_{i=N_1+1}^{N} \gamma_\varepsilon^i \\
&\vec{u}_\varepsilon(x, t+T) = u_\varepsilon(x,t)
\end{aligned}
\right. \qquad (1)
$$

$\vec{g}_i$ compact support functions on $\partial\omega_i$

T-periodic in time

$$\sum_i \int_{\gamma_\varepsilon^i} \vec{g}_i \cdot \vec{n} = 0$$

$\varepsilon \ll 1$

or

$$\begin{cases} \dfrac{\partial \vec{u}_\varepsilon}{\partial t} - \nu \Delta \vec{u}_\varepsilon + (\vec{u}_\varepsilon, \nabla)\vec{u}_\varepsilon + \nabla p = 0 \\ \qquad\qquad\qquad x \in B_\varepsilon,\ t \in \mathbb{R}, \\ \operatorname{div} \vec{u}_\varepsilon = 0 \\ \vec{u}_\varepsilon = \vec{g}_i\left(\dfrac{x - O_i}{\varepsilon}, \dfrac{t}{\varepsilon^2}\right),\ x \in \gamma_\varepsilon^i \\ \vec{u}_\varepsilon = 0,\ x \in \partial B_\varepsilon \setminus \gamma_\varepsilon \\ \vec{u}_\varepsilon(x, t + \varepsilon^2 T) = \vec{u}_\varepsilon(x, t) \end{cases} \qquad (1_2)$$

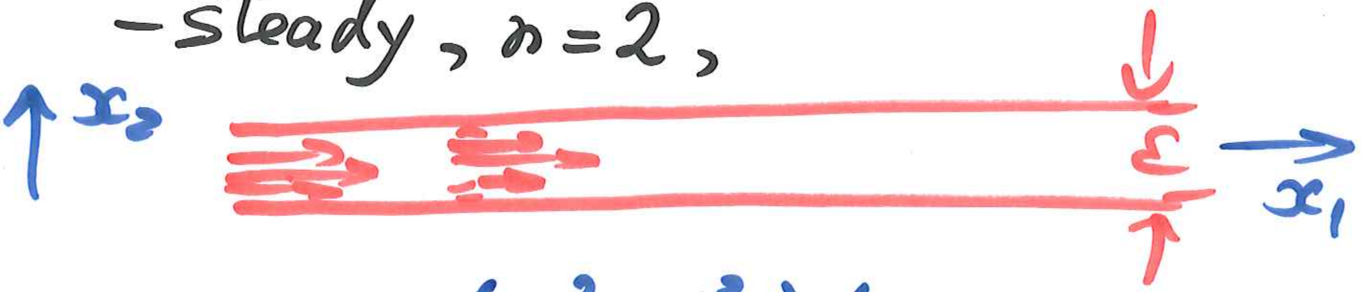
or steady problem

$$\begin{cases} -\nu \Delta \vec{u}_\varepsilon + (\vec{u}_\varepsilon, \nabla)\vec{u}_\varepsilon + \nabla p = 0 \\ \qquad\qquad\qquad x \in B_\varepsilon, \\ \operatorname{div} \vec{u}_\varepsilon = 0, \\ \vec{u}_\varepsilon = \vec{g}_i\left(\dfrac{x - O_i}{\varepsilon}\right),\ x \in \gamma_\varepsilon^i, \end{cases} \qquad (1_3)$$

Complete as. expansion is built.

1. For every edge the Poiseuille flow is built

- steady, $n=2$,



$$u_1 = -a\left(x_2^2 - \varepsilon^2/4\right)/2\nu,\quad u_2 = 0,$$

$$p = -a x_1 + b$$

- steady, $n=3$,

$$u_1(x_2, x_3) = \nu^{-1} a\, U_p,\quad U_p : -\Delta_\perp U_p = 1$$

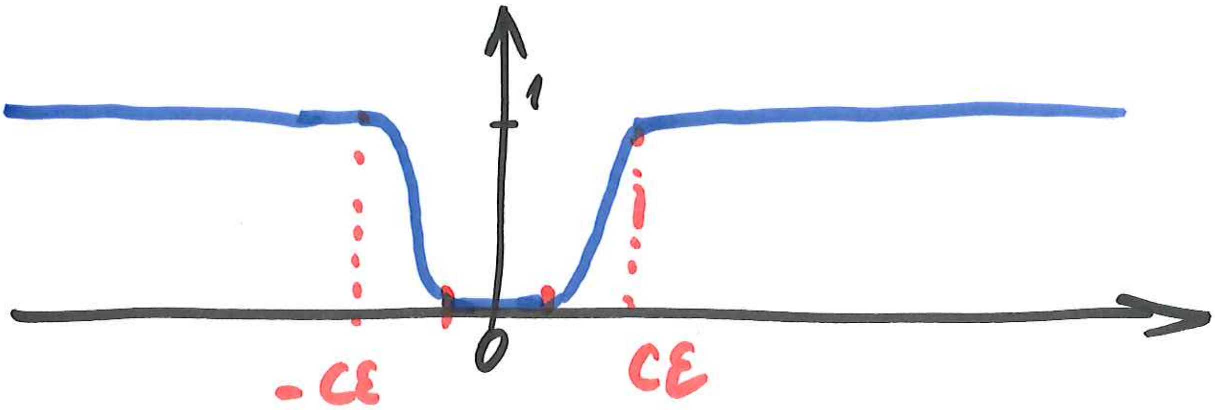
$u_2 = u_3 = 0,$ on σ_ε, $U_p|_{\partial\sigma_\varepsilon} = 0$

$p = -a x_1 + b$

- non-steady: $u_1(x_2, x_3, t)$, $u_2 = u_3 = 0$,

$p = -a x_1 + b$.

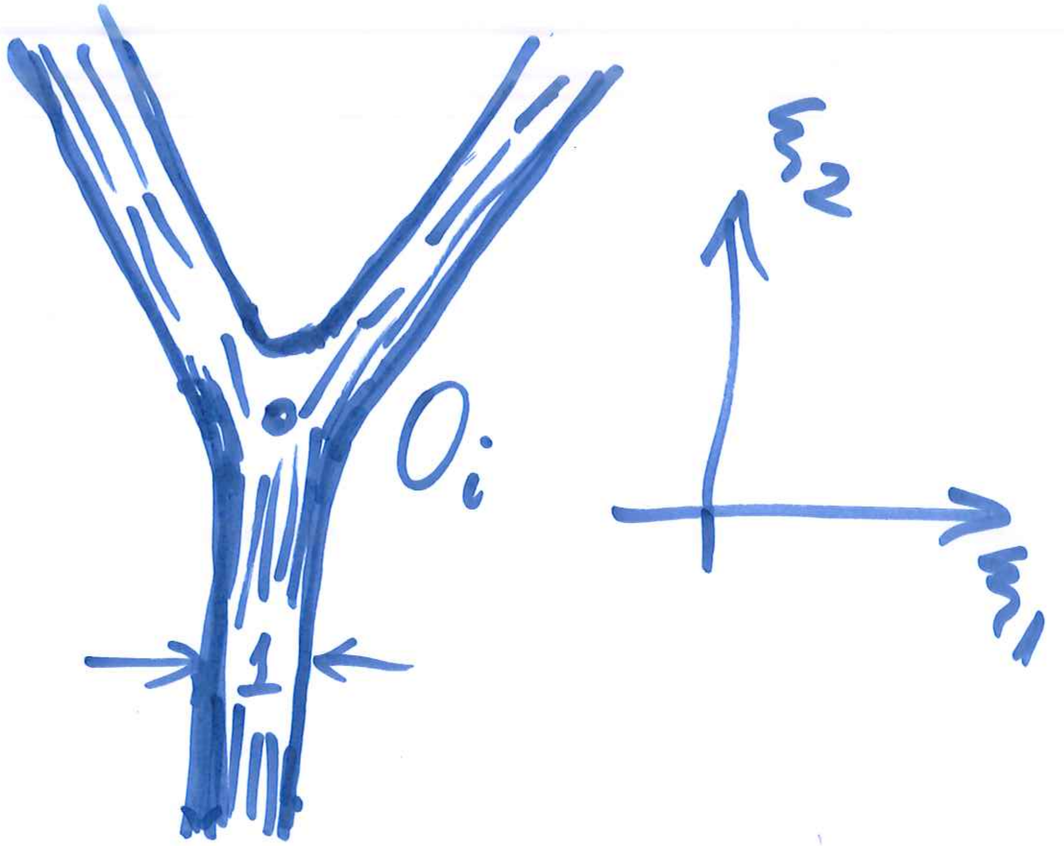
2. Each Poiseuille is multiplied by a smooth cut-of function vanishing at some neighborhood of the nodes, their diam $= O(\varepsilon)$



$\Downarrow\Downarrow$

Residual with support in $O(\varepsilon)$ neighborhood of the nodes.

3. For every node O_i we get the Stokes eq. in dilated variables $\xi = x/\varepsilon$



in an unbounded domain independent of ε

4. These problems are related by a problem for the pressure on the graph.

5. Equations on the graph.

5.1 Steady state.

Find $p \in H^1(\mathcal{B})$

$$-\frac{\partial^2}{\partial x_1^{(e)2}} p(x_1^{(e)}) = 0, \quad \text{(linearity)}$$

$$x_1^{(e)} \in (0, |e|) \quad \forall e$$

$$-\sum_{e:\, O_l \in e} \varkappa^{(e)} \frac{\partial p}{\partial x_1^{(e)}}(0) = 0, \quad \text{(flux conservation)}$$

$$\forall O_l,\ l = 1, \dots, N_1$$

$$-\varkappa^{(e)} \frac{\partial p}{\partial x_1^{(e)}}(0) = \int_{\gamma_\varepsilon^i} \vec{g}_i \cdot \vec{n}\, ds, \quad i = N_1+1, \dots, N.$$

$$\varkappa^{(e)} = \int_{\sigma_\varepsilon^{(e)}} U_p \, dx$$ – coef. prop. in the law flux–pressure slope.

For higher order approximations: given pressure jumps in the nodes.

5.2. Time dependent equation on the graph

Operator $L^{(e)}$ relating the pressure drop and the flux in an infinite tube

Given $S \in L^2_{per}(0,T)$ find

$\mathcal{V} \in L^2_{per}(0,T; H^1_0(\sigma^{(e)}))$,

$\frac{\partial \mathcal{V}}{\partial \tau} \in L^2_{per}(0,T; L^2(\sigma^{(e)}))$ s.t.

$$\begin{cases} \frac{\partial \mathcal{V}}{\partial \tau}(y^{(e)\prime}, \tau) - \nu \Delta' \mathcal{V}(y^{(e)\prime}, \tau) = S(\tau), \\ \qquad y^{(e)\prime} \in \sigma^{(e)}, \ \tau \in \mathbb{R}, \qquad (2) \\ \mathcal{V}(y^{(e)\prime}, \tau)\big|_{\partial\sigma^{(e)}} = 0 \end{cases}$$

denote

$$L^{(e)} S(\tau) = \int_{\sigma^{(e)}} \mathcal{V}(y^{(e)\prime}, \tau)\, dy^{(e)\prime}.$$

$L^{(e)} : L^2_{per}(0,T) \longleftrightarrow H^1_{per}(0,T)$

Problem on the graph

Given $\Psi_\ell \in H^1_{per}(0,T)$, $\ell = 1, \dots, N$,

$F^{(e_i)} \in H^1_{per}(0,T; L^2(\mathcal{B}))$, $i = 1, \dots, M$

find $p \in L^2_{per}(0,T; H^1(\mathcal{B}))$ s.t.

$$
\begin{cases}
\dfrac{\partial}{\partial x_1^{(e)}} \left(L^{(e)} \dfrac{\partial p}{\partial x_1^{(e)}} (x_1^{(e)}, \tau) \right) = F^{(e_i)}(x_1^{(e)}, \tau), \\
\qquad x_1^{(e)} \in (0, |e|), \\
\qquad \forall e = e_j, \ j = 1, \dots, M \\
- \displaystyle\sum_{e : O_\ell \in e} \left(L^{(e)} \dfrac{\partial p^{(e)}}{\partial x_1^{(e)}} \right)(0, \tau) = \Psi_\ell(\tau), \\
\qquad \ell = 1, \dots, N_1, \\
- L^{(e)} \dfrac{\partial p^{(e)}}{\partial x_1^{(e)}} (0, \tau) = \Psi_\ell(\tau), \\
\qquad \ell = N_1 + 1, \dots, N
\end{cases}
\tag{3}
$$

$$
\boxed{\sum_{i=1}^{M} \int_0^{|e_i|} F^{(e_i)} \, d x_1^{(e_i)} + \sum_{\ell=1}^{N} \Psi_\ell(\tau) = 0} \tag{V}
$$

compatibility cond.

Th. 1. Let $\mathcal{H} \in H^1_{per}(0,T)$.

$\exists!$ (V, S) satsfying (2) and

$$\int_{\sigma^{(e)}} V(y^{(e)'}, \tau)\, dy^{(e)'} = \mathcal{H}(\tau)$$

$$\|V\|_{L^2(0,T;\, H^1_0(\sigma^{(e)}))} + \left\|\frac{\partial V}{\partial \tau}\right\|_{L^2(0,T;\, L^2(\sigma^{(e)}))}$$

$$+ \|S\|_{L^2(0,T)} \leq c\, \|\mathcal{H}\|_{H^1(0,T)}$$

and $\exists\, C_e > 0$:

$\forall\, Q \in L^2_{per}(0,T)$

$$C_e^{-1} \|Q\|_{L^2(0,T)} \leq \|L^{(e)} Q\|_{H^1(0,T)} \leq$$

(Galdi)

$$\leq C_e \|Q\|_{L^2(0,T)}$$

Th2. Let $\langle \cdot \rangle = \frac{1}{T} \int_0^T \cdot \, d\tau$.

$$\langle L^e S \rangle = \varkappa^e \langle S \rangle,$$

$\varkappa^{(e)} = \int_{\sigma^{(e)}} U(y')\, dy' > 0$, U :

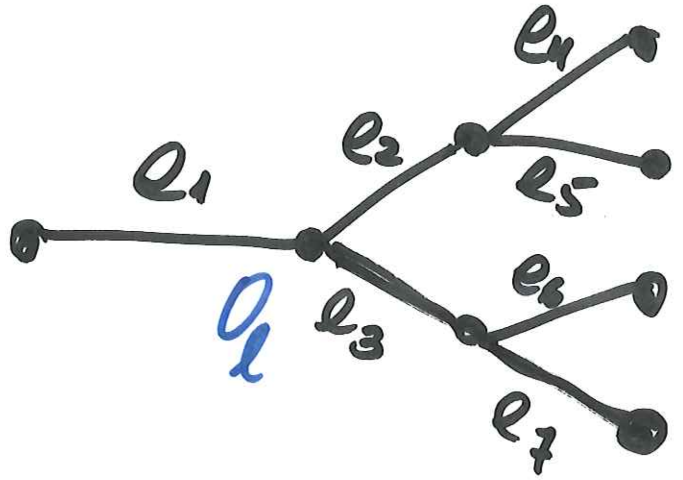
$-\nu \Delta U = 1$, $y' \in \sigma^{(e)}$, $U|_{\partial\sigma^{(e)}} = 0$.

Th. 3. Pb. (3) admits a unique (up to an additive function of τ from $L^2_{per}(0,T)$) solution p.

<u>Proof</u>. Lax-Milgram argument.

<u>Conclusion</u> : new Reynolds' type equation on the graph, non-local in time.

6. "Randomnization"



$$
\begin{cases}
p \text{ linear} \\
-\sum\limits_{e:\,O_\ell \in e} \varkappa^{(e)} \dfrac{\partial p}{\partial x_1^{(e)}}(0) = 0, \ \forall \ell = 1,\dots,N_1 \\
p\big|_{e_i}^{(0)} = p\big|_{e_j}^{(0)} + \Delta p_{ij}, \quad i \neq j, \quad O_\ell \in e_i, e_j \\
-\varkappa^{(e)} \dfrac{\partial p}{\partial x_1^{(e)}}(0) = \int\limits_{\gamma_\varepsilon^i} \vec{g}_i \cdot \vec{n}\, ds
\end{cases}
$$

Δp_{ij} random variables (indep.)

Eq. for expectation; deviations?

- Random sections $\sigma^{(i)}$

- Fractal extension: distribution of the fluxes in the edges of rank N $N \to +\infty$.

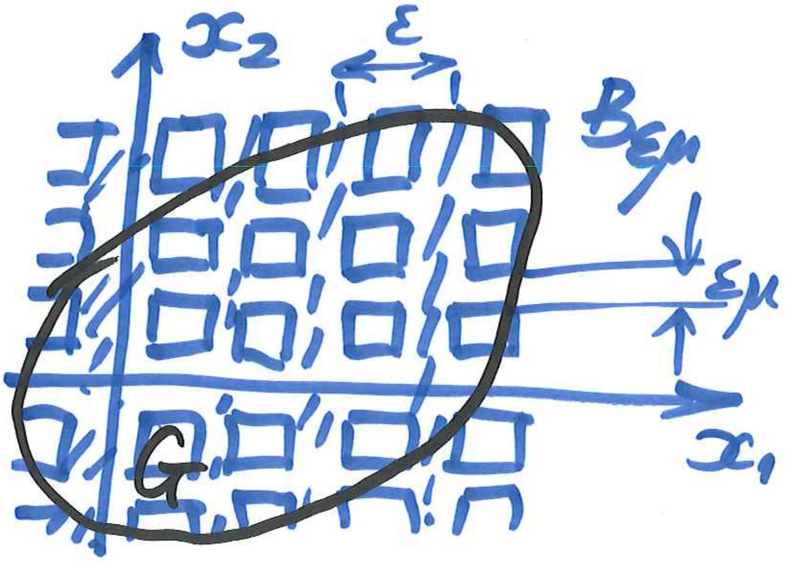
- Example of a random thin structure: Homogenization of a random lattice structure

G.P. Comp. Math. Math. Phys. (USSR) 1983, $\underline{23}$, N 5.

$G \subset \mathbb{R}^2$ bounded domain, $\partial G \in C^2$,

$$B_{\varepsilon\mu} = \bigcup_{k=-\infty}^{+\infty} \Big(\{(x_1,x_2)\in\mathbb{R}^2 \mid |x_2 - k\varepsilon| < \tfrac{\varepsilon\mu}{2}\}$$

$$\cup \{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1 - k\varepsilon| < \tfrac{\varepsilon\mu}{2}\}\Big)$$



$$B_k^j = \{(x_1,x_2)\in\mathbb{R}^2 \mid |x_{3-j} - k\varepsilon| < \tfrac{\varepsilon\mu}{2}\},\ j=1,2,$$

$\updownarrow$

$A_{B_k^j}$ random valued constant 2x2-matrix indep. in aggregate

$$P\{A_{B_k^j} = A^{(s)}\} = p_s,\quad s=1,\dots,2$$

$$\sum_{s=1}^{2} p_s = 1,\ A^{(s)} = A^{(s)T} > 0,\ A^{(s)} = (a_{ij}^{(s)})_{1\le i,j\le 2}$$

$$\Pi = \bigcup_{k,j=-\infty}^{\infty} (B_k^1 \cap B_j^2) : A^{(0)} = A^{(0)T} > 0.$$

$$\begin{cases} -\operatorname{div}(A\nabla u_{\varepsilon\mu}) = f(x), & x\in B_{\varepsilon\mu}\cap G, \\ A\nabla u_{\varepsilon\mu}\cdot\vec{n} = 0, & x\in\partial B_{\varepsilon\mu}\cap G, \\ u_{\varepsilon\mu} = 0, & x\in\partial G\cap\overline{B}_{\varepsilon\mu}, \end{cases}$$

where $f\in C^1(\bar{G})$,

$$A = \begin{cases} A_{B_k^j} & \text{in } B_k^j\setminus\Pi, \\ A^{(0)} & \text{in } \Pi. \end{cases}$$

Denote $\bar{A} = (\bar{a}_{ij})_{1\le i,j\le 2}$,

$$\bar{a}_{11} = \sum_{s=1}^{2} p_s \frac{a_{11}^{(s)} - a_{12}^{(s)}\left(a_{22}^{(s)}\right)^{-1} a_{21}^{(s)}}{2}, \quad \bar{a}_{12} = 0,$$

$$\bar{a}_{22} = \sum_{s=1}^{2} p_s \frac{a_{22}^{(s)} - a_{21}^{(s)}\left(a_{11}^{(s)}\right)^{-1} a_{12}^{(s)}}{2}, \quad \bar{a}_{21} = 0,$$

$$u_0: \begin{cases} -\operatorname{div}(\bar{A}\nabla u_0) = f(x), & x\in G, \\ u_0|_{\partial G} = 0 \end{cases}$$

Th. $\forall\,\delta\in(0,1)$

$$P\left\{\frac{\|u_{\varepsilon\mu} - u_0\|_{L^2(B_{\varepsilon\mu}\cap G)}}{\sqrt{\operatorname{mes}(B_{\varepsilon\mu}\cap G)}} > \left(\sqrt{\varepsilon}+\sqrt{\mu}\right)^{1-\delta}\right\} \le C\varepsilon^{\delta}.$$