Efficient sampling of random colorings

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Random Graphs and its Application for Complex Networks

Université Jean Monnet, Saint Etienne - 4th October, 2018

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joint work with Michelle Delcourt and Luke Postle.

Counting proper colourings

G: graph on n vertices and maximum degree Δ

k: positive integer, number of colours.

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Q: Is it possible to approximate $|\Omega_k(G)|$ efficiently?

A fully polynomial-time randomised approximation scheme (FPRAS), is a randomised algorithm that runs in polynomial time in n and $1/\delta$ whose output is $(1 \pm \delta)|\Omega_k(G)|$ with "high" probability.

A fully polynomial-time almost uniform sampler (FPAUS), is a randomised algorithm that runs in polynomial time in n that generates an element of $\Omega_k(G)$ according to a probability distribution that is "close" to uniform.

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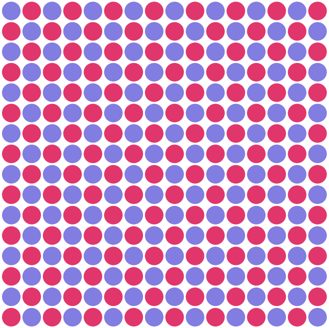

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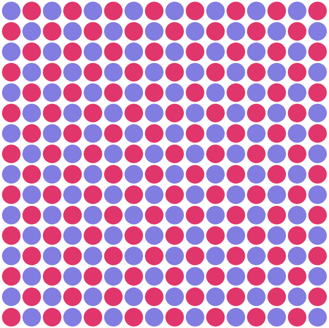

Random colourings and Glauber dynamics

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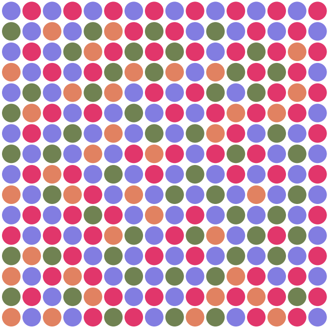

Remarks:

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Conjecture (Folklore)

If $k \geqslant \Delta + 2$, Glauber dynamics for k-colourings of G on n vertices satisfies

$$t_{\mathrm{Glau}}(\epsilon) = n^{O(1)} \qquad \text{(stronger version: } O(n \log n)) .$$

Previous results

Conjecture: If $k \geqslant \Delta + 2$, then $t_{\mathrm{Glau}} = n^{O(1)}$ (stronger: $O(n \log n)$).

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Better bounds for classes of graphs:

- Large girth
- Planar graphs
- Trees
- Bounded treewidth
- Erdős-Rényi random graphs

Path coupling

Bubley and Dyer (1997):

- Define pre-metric (Γ, ω) with $V(\Gamma) = \Omega_k(G)$ and $\omega : E(\Gamma) \to [0,1]$.
- Let d be the metric induced by (Γ, ω) on $\Omega_k(G)$ using minimum weight paths.
- Define a coupling $(X_t, X'_t) \to (X_{t+1}, X'_{t+1})$ for $X_t X'_t \in E(\Gamma)$.
- If there exists $\alpha > 0$ such that for every $X_t X'_t \in E(\Gamma)$

$$\mathbb{E}[\nabla(d)] := \mathbb{E}[d(X_{t+1}, X'_{t+1}) - d(X_t, X'_t)] < -\alpha ,$$

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Obstacle: atypical pairs (X_t, X'_t).

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Both proofs also extend to list-colourings (only $k \geqslant 2\Delta$ was known).

Kempe components and Flip dynamics

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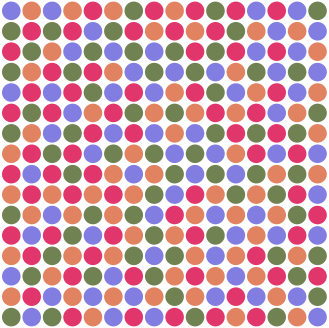

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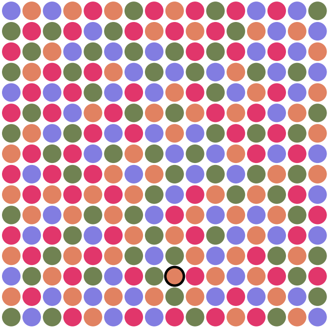

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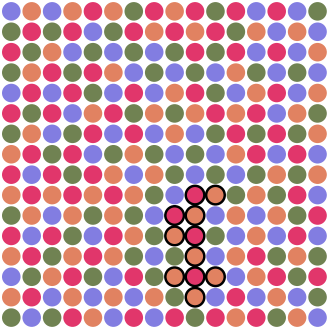

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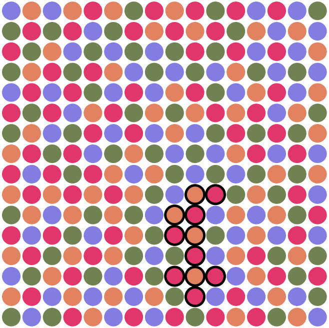

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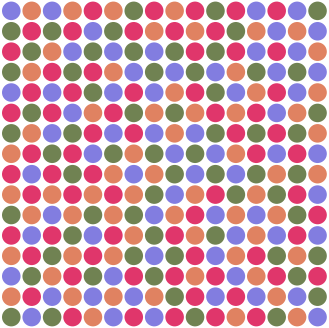

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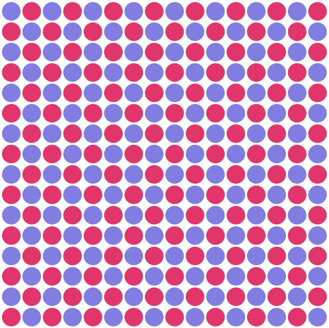

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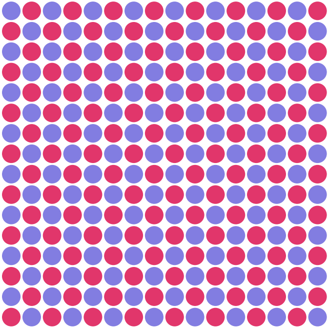

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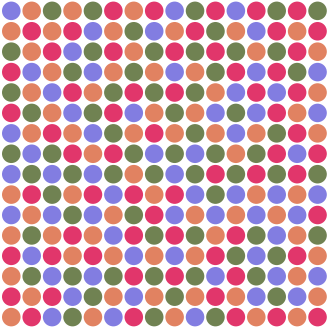

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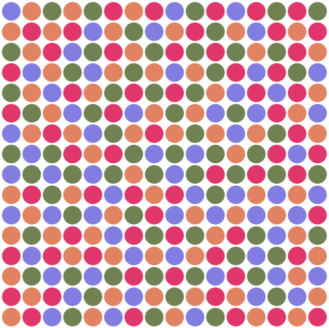
Also converges to a uniform colouring



Kempe component: given $\sigma \in \Omega_k(G)$, $u \in V(G)$, $c \in [k]$

$$S_\sigma(u,c) = \{v \in V(G) : (\sigma(u), c)\text{-alternating path between } u \text{ and } v\}$$

Flip dynamics: Markov chain (Y_t) governed by $\mathbf{p} = (p_1, p_2, \dots)$ with state space $\Omega_k(G)$ and transitions defined by

- Choose u *uar* in $V(G)$.
- Choose c *uar* in $[k]$.
- Let $S = S_\sigma(u,c)$ and $\ell = |S|$. With probability p_ℓ/ℓ, Y_{t+1} obtained from Y_t by flipping S; otherwise, $Y_{t+1} = \sigma$.

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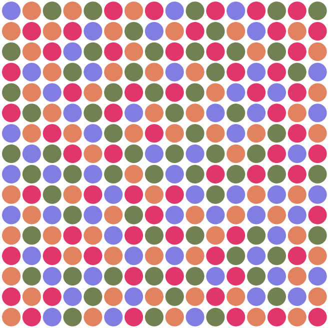
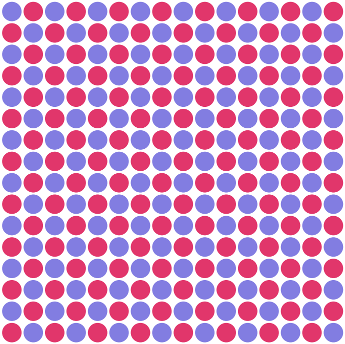
Mixing time: t_{flip}

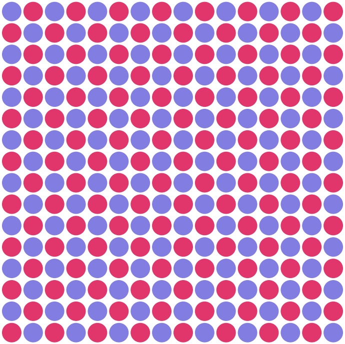


Glauber vs. flip dynamics

Glauber dynamics



Flip dynamics with $p_\ell = \ell$



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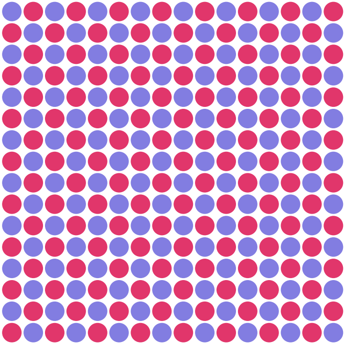

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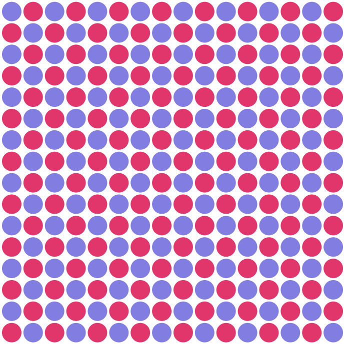

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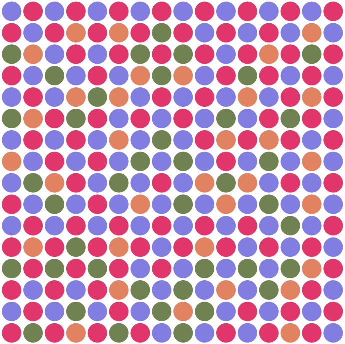

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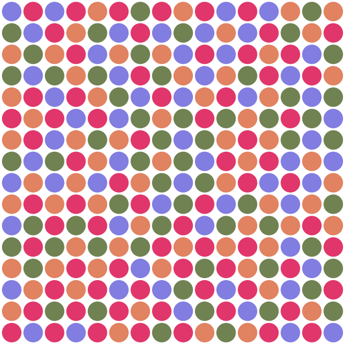

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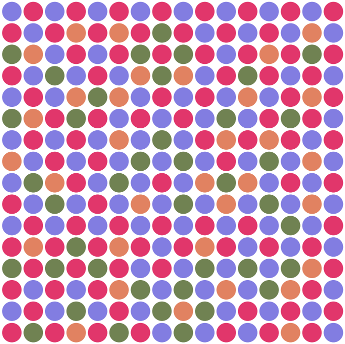

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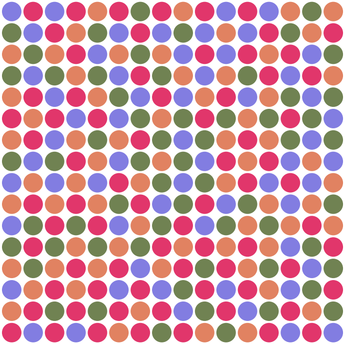

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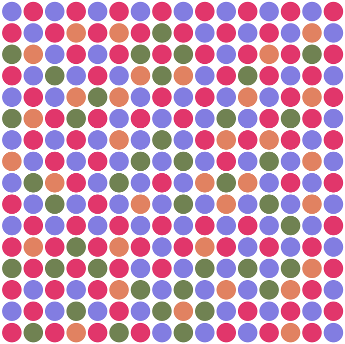

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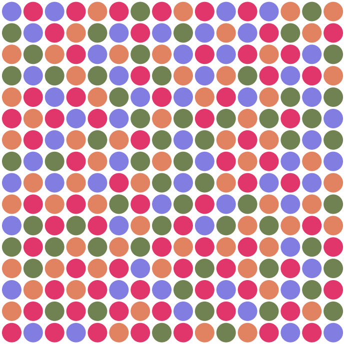

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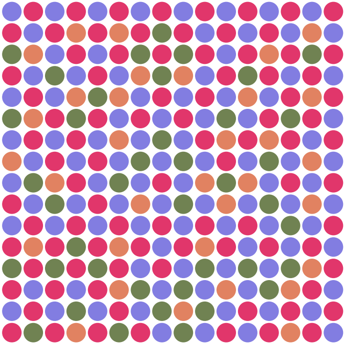

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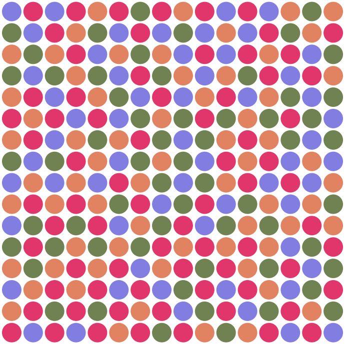

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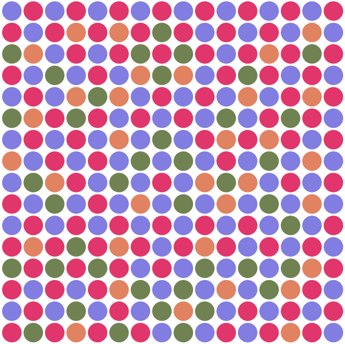

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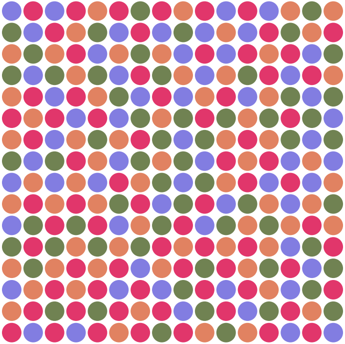

- If $p_1 = 1$, then flip dynamics embeds Glauber dynamics ($t_{\text{flip}} \leqslant t_{\text{Glau}}$).
- Even more, flip dynamics moves even if c appears in $N(v)$ ($t_{\text{flip}} \ll t_{\text{Glau}}$).
- Assume that $p_\ell = 0$, for $\ell \geqslant \ell_0$ (no big flips are allowed).
- **Vigoda (1999)**: If $k > (1+\epsilon)\Delta$, one can simulate each move in flip dynamics with $O(1)$ moves in Glauber dynamics:

$$t_{\text{Glau}} = O(n \log k \cdot t_{\text{flip}})$$

Recall: Path coupling

Bubley and Dyer (1997):

- Define pre-metric (Γ, ω) with $V(\Gamma) = \Omega_k(G)$ and $\omega : E(\Gamma) \to [0,1]$.
- Let d be the metric induced by (Γ, ω) on $\Omega_k(G)$ using minimum weight paths.
- Define a coupling $(Y_t, Y'_t) \to (Y_{t+1}, Y'_{t+1})$ for $Y_t Y'_t \in E(\Gamma)$.
- If there exists $\alpha > 0$ such that for every $Y_t Y'_t \in E(\Gamma)$

$$\mathbb{E}[\nabla(d)] := \mathbb{E}[d(Y_{t+1}, Y'_{t+1}) - d(Y_t, Y'_t)] < -\alpha \ ,$$

then

$$t_{\mathrm{mix}} = O(\alpha^{-1} \log n) \ .$$

Obstacle: atypical pairs (Y_t, Y'_t).

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Obstacle: atypical pairs (Y_t, Y'_t).

Ways to overcome it:

- *Burn-in method*: multi-step analysis, run the chains until we reach a typical pair (**Chen, Moitra**).
- *Extremal metric*: single-step analysis with metric tailored to favour typical pairs (**Delcourt, P., Postle**).

Pre-metric (Γ, ω) and configurations

Γ: graph on $\Omega_k(G)$ where σ and τ are adjacent iff they only differ at a vertex v.

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(σ, τ) has an r-configuration $(a_1, \dots, a_r; b_1, \dots, b_r)$ for c if

- $N(v) \cap \sigma^{-1}(c) = \{w_1, \dots, w_r\}$;
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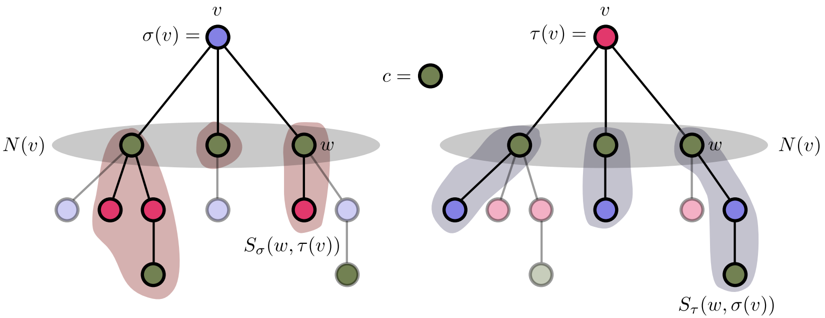

3-configuration $(2, 2, 3; 4, 1, 2)$

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A configuration is extremal if

- $r = 1$ and it is either $(2; 1)$ or $(1; 2)$;
- $r = 2$ and it is either $(3, 3; 1, 1)$ or $(1, 1; 3, 3)$.

Let $\beta_{\sigma,\tau}$ be the proportion of vertices $w \in N(v)$ such that $\sigma(w) = c$ and the configuration for c is extremal. For $\gamma > 0$ sufficiently small,

$$\omega(\sigma, \tau) := 1 - \gamma(1 - \beta_{\sigma,\tau})$$

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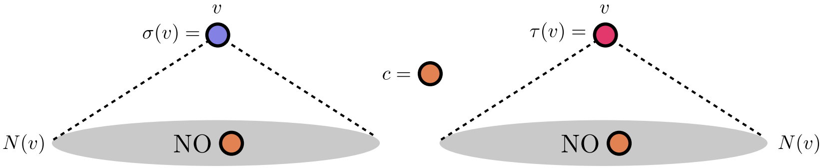
We consider the pre-metric (Γ, ω), that extends to a metric d.

$$d(\sigma, \tau) = d_H(\sigma, \tau) - d_B(\sigma, \tau)$$

where d_H is the Hamming distance. If $\gamma = 0$, then $d = d_H$ (**Vigoda, 1999**).

Given σ, τ that differ only at v and $c \in [k]$:

- **GOOD**: If $r = 0$, use identity coupling and d_H changes by $f(\emptyset; \emptyset) = -1$.



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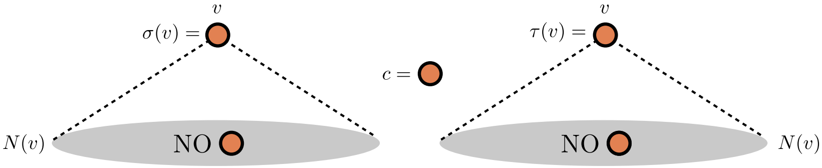

Vigoda's coupling and change of d_H

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- **GOOD**: If $r = 0$, use identity coupling and d_H changes by $f(\emptyset; \emptyset) = -1$.
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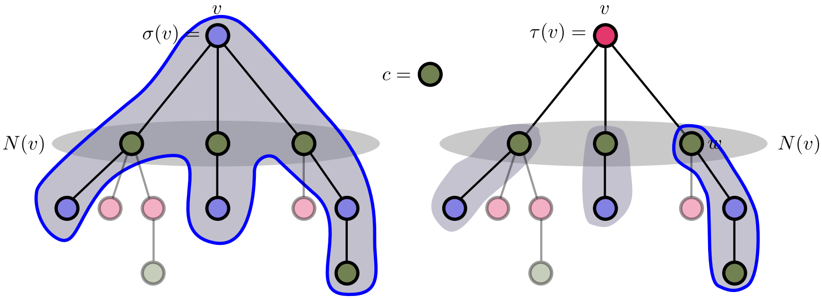

(1)

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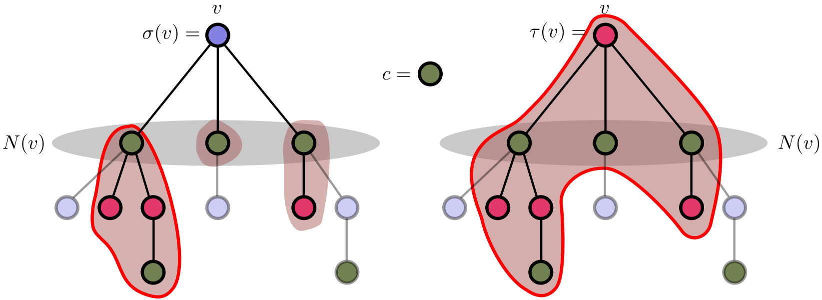

(2)

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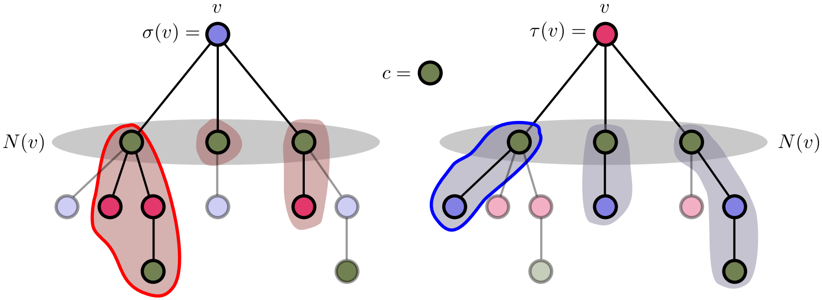

(3.1)

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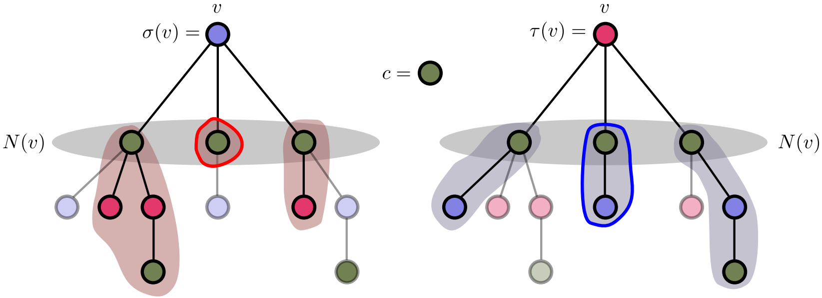

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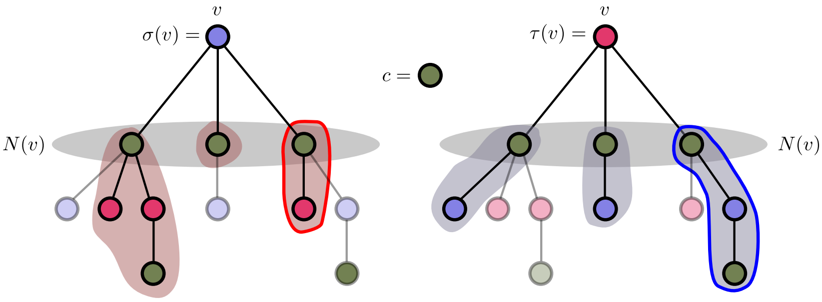

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$$f(a_1, \dots, a_r; b_1, \dots, b_r) = \underbrace{\Big(\sum_{i \in [r]} a_i - a_{\max}\Big) p_a}_{(1)} + \underbrace{\Big(\sum_{i \in [r]} b_i - b_{\max}\Big) p_b}_{(2)} + \sum_{i \in [r]} \underbrace{\left(a_i q_i + b_i q_i' - \min\{q_i, q_i'\}\right)}_{(3.i)} .$$

Vigoda's coupling and change of d_H

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Under the Hamming metric $d = d_H$ ($\gamma = 0$), if we have

$$f(a_1,\dots,a_r;b_1\dots,b_r) \leqslant \kappa r - 1\ .$$

then we get rapid mixing for $k \geqslant \kappa\Delta + 1$

$$\mathbb{E}[\nabla(d_H)] \leqslant \frac{1}{kn}\sum_{c\in[k]} f(\text{configuration for } c) \leqslant \frac{1}{kn}(\kappa|N(v)| - k) < -\frac{1}{kn} =: -\alpha\ .$$

Minimise: κ

subject to:

$$f(a_1,\dots,a_r;b_1\dots,b_r)\leqslant \kappa r-1 \qquad \text{for every conf.}(a_1,\dots,a_r;b_1,\dots,b_r)$$
$$p_1=1$$
$$p_i-p_{i+1}\leqslant 0 \qquad \text{for } i\geqslant 1$$
$$p_i\geqslant 0 \qquad \text{for } i\geqslant 1$$

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$$
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f(a_1,\ldots,a_r;b_1\ldots,b_r) &\leqslant \kappa r - 1 && \text{for every conf.}(a_1,\ldots,a_r;b_1,\ldots,b_r)\\
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Number of variables and constraints depends on Δ.

Relax it to a finite LP, with same solution set.

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Constraints for extremal configurations:

$$f(2;1) = p_1 - p_3 \leqslant \kappa - 1$$
$$f(3,3;1,1) = 2p_1 + 4p_3 - p_7 \leqslant 2\kappa - 1$$

Since $p_1 = 1$ and $p_7 \geqslant 0$, it follows that $\kappa \geqslant 11/6$.

Moreover, if $\kappa = 11/6$, then $p_3 = 1/6$ and $p_7 = 0$.

Flip probabilities and expected change of d_H

Vigoda: $\kappa = 11/6$, $\mathbf{p}_{\mathrm{Vig}} = \left(1, \frac{13}{42}, \frac{1}{6}, \frac{2}{21}, \frac{1}{21}, \frac{1}{84}, 0, 0, \dots\right)$, optimal solution,

$$f(a_1, \dots, a_r; b_1, \dots, b_r) \leqslant \frac{11}{6} r - 1 \Rightarrow \mathbb{E}(\nabla(d_H)) \leqslant \frac{1}{kn} \left(\frac{11}{6}\Delta - k\right) \leqslant -\alpha \, .$$

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$(a_1, \dots, a_r; b_1, \dots, b_r)$ is **p**-extremal if $f(a_1, \dots, a_r; b_1, \dots, b_r) = \frac{11}{6} r - 1$.

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- there are 6 $\mathbf{p}_{\mathrm{Vig}}$-extremal configurations: $(2;1)$, $(3;1)$, $(4;1)$, $(5;1)$, $(2,2;1,1)$ and $(3,3;1,1)$.

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Let $\epsilon = \frac{11}{6} - \frac{161}{88} \approx 0.00378$. The expected change of d_H is

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  $$kn\,\mathbb{E}(\nabla(d_B)) \geqslant C\gamma\beta_{\sigma,\tau}\Delta\,,$$

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Conclusions

Conjecture (Folklore)

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