Poisson-Voronoi graph on a Riemannian manifold

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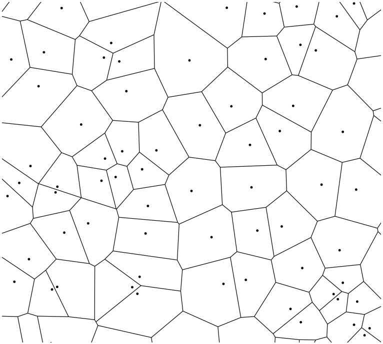
Poisson-Voronoi tessellation in $\mathbb{R}^n$



- $\mathcal{P}_\lambda$ homogeneous Poisson point process in $\mathbb{R}^n$ of intensity λ
- For every nucleus $x \in \mathcal{P}_\lambda$, associated cell

$$C(x, \mathcal{P}_\lambda) := \{y \in \mathbb{R}^n : \|y - x\| \leq \|y - x'\| \ \forall x' \in \mathcal{P}_\lambda\}$$

Typical Poisson-Voronoi cell

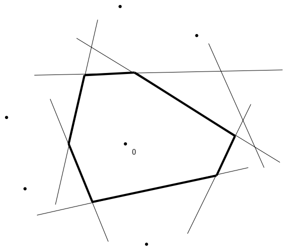
- Typical cell $\mathcal{C}$ *chosen uniformly among all cells*

$$\mathbb{E}(f(\mathcal{C})) := \lim_{r\to\infty} \frac{1}{N_r} \sum_{x\in\mathcal{P}_\lambda\cap\mathcal{B}^{(\mathbb{R}^n)}(o,r)} f(C(x,\mathcal{P}_\lambda)) \text{ a.s.}$$

$$\mathbb{E}(f(\mathcal{C})) := \frac{1}{\lambda \mathrm{vol}^{(\mathbb{R}^n)}(B)} \mathbb{E}\left(\sum_{x\in\mathcal{P}_\lambda\cap B} f(C(x,\mathcal{P}_\lambda)) \right), \; B\in\mathcal{B}(\mathbb{R}^n)$$

for every translation-invariant functional f

- Theorem (Slivnyak) $\mathcal{C} \overset{D}{=} C(o, \mathcal{P}_\lambda \cup \{o\})$



Mecke's formula

- For any $\ell \geq 1$ and any non-negative measurable function f,

$$\mathbb{E}\Bigg(\sum_{\{x_1,\dots,x_\ell\}\subset \mathcal{P}_\lambda} f(\{x_1,\dots,x_\ell\},\mathcal{P}_\lambda)\Bigg) = \frac{\lambda^\ell}{\ell!}\int \mathbb{E}(f(\{x_1,\dots,x_\ell\},\mathcal{P}_\lambda \cup \{x_1,\dots,x_\ell\}))\mathrm{d}x_1\dots\mathrm{d}x_\ell$$

- Characterization of the Poisson point process

Two mean calculations in $\mathbb{R}^2$

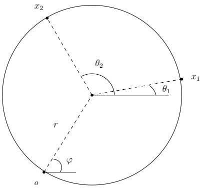
Mean area

$$\mathbb{E}(\mathrm{vol}^{(\mathbb{R}^2)}(\mathcal{C})) = \int_{\mathbb{R}^2} \mathbb{P}(x \in C(o, \mathcal{P}_\lambda \cup \{o\}) \mathrm{d}x = 2\pi \int_0^\infty e^{-\lambda \pi r^2} r \mathrm{d}r = \frac{1}{\lambda}$$

Mean number of vertices

$\mathcal{N}(\cdot) :=$ number of vertices, $B(x, y, z) :=$ circumscribed disk of the triangle xyz

$$\begin{aligned}
\mathbb{E}(\mathcal{N}(\mathcal{C})) &= \mathbb{E} \sum_{\{x_1, x_2\} \subset \mathcal{P}_\lambda} \mathbf{1}_{\{B(o, x_1, x_2) \cap \mathcal{P}_\lambda = \emptyset\}} \\
&= \frac{\lambda^2}{2} \int \mathbb{P}(B(o, x_1, x_2) \cap \mathcal{P}_\lambda = \emptyset) \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{\lambda^2}{2} \int e^{-\lambda \mathrm{vol}^{(\mathbb{R}^2)}(B(o, x_1, x_2))} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= 4\pi\lambda^2 \int e^{-\lambda \pi r^2} r^3 \mathrm{d}r \iint_{(0, 2\pi)^2} \sin(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}) | \sin(\frac{\theta_2 - \theta_1}{2}) | \mathrm{d}\theta_1 \mathrm{d}\theta_2 \\
&= 6
\end{aligned}$$



Poisson-Voronoi tessellation on a Riemannian manifold

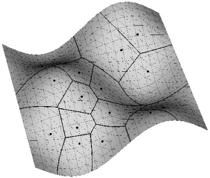
Mean asymptotics for the number of vertices of a Voronoi cell

Limit theorems for the empirical mean of the number of vertices

Joint work with
Aurélie Chapron (Paris Nanterre) & **Nathanaël Enriquez** (Paris-Sud)

Plan

Model



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- M Riemannian manifold of dimension n endowed with the distance $d^{(M)}$ and the volume measure $\mathrm{vol}^{(M)}$
- $\mathcal{P}_\lambda$ Poisson point process in M of intensity measure $\lambda\mathrm{vol}^{(M)}$
- Voronoi cell associated with $x \in \mathcal{P}_\lambda$

$$C^{(M)}(x, \mathcal{P}_\lambda) = \{y \in M : d^{(M)}(x, y) \leq d^{(M)}(x', y)\, \forall x' \in \mathcal{P}_\lambda\}$$

Aim Study the mean geometrical characteristics of

$$\mathcal{C}^{(M)}_{x_o,\lambda} := C^{(M)}(x_0, \mathcal{P}_\lambda \cup \{x_0\}), \quad x_0 \in M \text{ fixed}$$

Context

- Influence of the local geometry of M around x_0 on the geometry of $C^{(M)}(x_0, \mathcal{P}_\lambda \cup \{x_0\})$

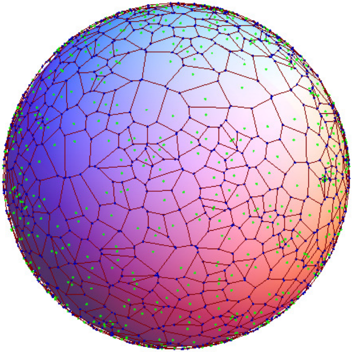
  $\rightsquigarrow$ *Need to do estimations at high intensity*

- Conversely, information provided by the tessellation on the geometry of M, both local (curvatures at x_0) and global (Gauss-Bonnet)

  $\rightsquigarrow$ *Need to show limit theorems in order to do a statistical estimation*

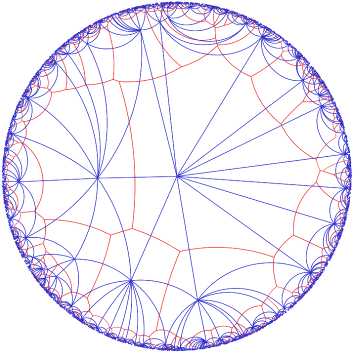
Previous works

$\mathcal{S}_k^2$
2-sphere of radius $1/k$



© **J. Diaz-Polo & I. Garay** (2014)

$\mathcal{H}_k^2$
hyperbolic plane of curvature $-k^2$



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Previous works

$\mathcal{S}_k^2$
2-sphere of radius $1/k$

$\mathcal{H}_k^2$
hyperbolic plane of curvature $-k^2$

Mean area

$$\mathbb{E}(\mathrm{vol}^{(\mathcal{S}_k^2)}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{S}_k^2)})) = \frac{1}{\lambda}\left(1 - e^{-\frac{4\pi\lambda}{k^n}}\right)$$

Mean number of vertices

$$\mathbb{E}\left(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{S}_k^2)})\right) = 6 - \frac{3k^2}{\pi\lambda} + e^{-\frac{4\pi\lambda}{k^2}}\left(6 + \frac{3k^2}{\pi\lambda}\right)$$

R. E. Miles (1971)

Mean area

$$\mathbb{E}(\mathrm{vol}^{(\mathcal{H}_k^n)}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{H}_k^2)})) = \frac{1}{\lambda}$$

Mean number of vertices

$$\mathbb{E}\left(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(\mathcal{H}_k^2)})\right) = 6 + \frac{3k^2}{\pi\lambda}.$$

Y. Isokawa (2000)

Main result

General assumptions on M: sectional curvatures bounded, global injectivity radius...

$\mathrm{Sc}_{x_0}^{(M)} :=$ scalar curvature of M at x_0

$$\mathbb{E}(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(M)})) \underset{\lambda\to\infty}{=} e_n - d_n \, \mathrm{Sc}_{x_0}^{(M)} \frac{1}{\lambda^{\frac{2}{n}}} + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

where $e_n = \mathbb{E}(\mathcal{N}(C^{(\mathbb{R}^n)}(o, \mathcal{P}_\lambda \cup \{o\})))$ and d_n are explicit.

$$\mathbb{E}(\mathrm{vol}^{(M)}(\mathcal{C}_{x_0,\lambda}^{(M)})) \underset{\lambda\to\infty}{=} \frac{1}{\lambda} + o\left(\frac{1}{\lambda^{1+\frac{2}{n}}}\right)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) \stackrel{?}{=} \frac{1}{4\pi} \int_M \mathrm{Sc}_x^{(M)} \mathrm{dvol}^{(M)}(x)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = F - E + V$$

by Euler's formula applied to the Poisson-Voronoi graph
F:= # faces, E:= # edges, V:= # vertices

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M) :=$ Euler characteristic of M

$$\chi(M) = F - E + V$$

$2E = 3V$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = F - \frac{1}{2}V$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \mathbb{E}(F) - \frac{1}{2}\mathbb{E}(V)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \lambda \mathrm{vol}^{(M)}(M) - \frac{1}{2}\mathbb{E}(V)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \lambda \mathrm{vol}^{(M)}(M) - \frac{1}{6}\mathbb{E}\left(\sum_{x\in\mathcal{P}_\lambda} \mathcal{N}(\mathcal{C}_{x,\lambda}^{(M)})\right)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \lambda \mathrm{vol}^{(M)}(M) - \frac{\lambda}{6} \int \mathbb{E}(\mathcal{N}(\mathcal{C}_{x,\lambda}^{(M)}))\mathrm{d}\mathrm{vol}^{(M)}(x)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \lambda \mathrm{vol}^{(M)}(M) - \frac{\lambda}{6} \int \mathbb{E}(\mathcal{N}(\mathcal{C}_{x,\lambda}^{(M)})) \mathrm{d}\mathrm{vol}^{(M)}(x)$$

$$\boxed{6 - \frac{3\mathrm{Sc}_x^{(M)}}{2\pi\lambda} + o\left(\frac{1}{\lambda}\right)}$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \lambda \mathrm{vol}^{(M)}(M) - \lambda \mathrm{vol}^{(M)}(M) + \frac{1}{4\pi}\int \mathrm{Sc}_x^{(M)} \mathrm{vol}^{(M)}(x) + o(1)$$

Probabilistic proof of the Gauss-Bonnet theorem

M 2-dimensional compact manifold, $\chi(M)$:= Euler characteristic of M

$$\chi(M) = \frac{1}{4\pi} \int \mathsf{Sc}_x^{(M)} \mathrm{d}\mathsf{vol}^{(M)}(x)$$

Main result

General assumptions on M: sectional curvatures bounded, global injectivity radius...

$\mathrm{Sc}_{x_0}^{(M)} :=$ scalar curvature of M at x_0

$$\mathbb{E}(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(M)})) \underset{\lambda\to\infty}{=} e_n - d_n \, \mathrm{Sc}_{x_0}^{(M)} \frac{1}{\lambda^{\frac{2}{n}}} + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

where $e_n = \mathbb{E}(\mathcal{N}(C^{(\mathbb{R}^n)}(o, \mathcal{P}_\lambda \cup \{o\})))$ and d_n are explicit.

Sketch of proof: preliminary calculation

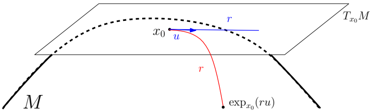
$$\mathbb{E}(\mathcal{N}(\mathcal{C}_{x_0,\lambda}^{(M)}))$$
$$= \mathbb{E} \sum_{\{x_1,\ldots,x_n\}\subset\mathcal{P}_\lambda} \sum_{\mathcal{B} \text{ circumball of } x_0,\ldots,x_n} \mathbf{1}_{\{\mathcal{B}\cap\mathcal{P}_\lambda=\emptyset\}}$$
$$\approx \frac{\lambda^n}{n!}\int_{\mathcal{B}^{(M)}(x_0,\varepsilon)^n} \mathbb{P}(\text{CircumBall}(x_0,\ldots,x_n)\cap\mathcal{P}_\lambda=\emptyset)\mathrm{d}\mathrm{vol}^{(M)}(x_1)\ldots\mathrm{d}\mathrm{vol}^{(M)}(x_n)$$
$$= \frac{\lambda^n}{n!}\int_{\mathcal{B}^{(M)}(x_0,\varepsilon)^n} e^{-\lambda\mathrm{vol}^{(M)}(\text{CircumBall}(x_0,\ldots,x_n))}\mathrm{d}\mathrm{vol}^{(M)}(x_1)\ldots\mathrm{d}\mathrm{vol}^{(M)}(x_n)$$

Aim Estimates when $x_1,\ldots,x_n\to x_0$ of

$$\mathrm{vol}^{(M)}(\text{CircumBall}(x_0,\ldots,x_n)) \quad \text{and} \quad \mathrm{d}\mathrm{vol}^{(M)}(x_1)\ldots\mathrm{d}\mathrm{vol}^{(M)}(x_n)$$

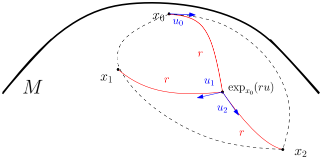
before applying Laplace's method

Sketch of proof: change of variables



Exponential map

For each unit-vector $u \in T_{x_0}M$, $\gamma : r \longmapsto \exp_{x_0}(ru)$ is the unique geodesic emanating from x_0 with speed 1 and $\gamma'(0) = u$.



Change of variables $\quad x_i = \exp_{|\exp_{x_0}(ru_0)}(ru_i),\ i = 1, \ldots, n$

Sketch of proof: asymptotics for the integrand

- Expansion of the Jacobian: Blaschke-Petkantschin formula

$$\mathrm{d}\mathrm{vol}^{(M)}(x_1)\ldots\mathrm{d}\mathrm{vol}^{(M)}(x_n)=\mathcal{J}\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}^{(\mathbb{S}^{n-1})}(u_0)\ldots\mathrm{d}\mathrm{vol}^{(\mathbb{S}^{n-1})}(u_n),$$

$$\mathcal{J}=n!\Delta(u_0,\ldots,u_n)\left(r^{n^2-1}-\frac{\sum_{i=0}^{n}\mathrm{Ric}_{x_0}^{(M)}(u_i)}{6}r^{n^2+1}+o(r^{n^2+1})\right)$$

$\mathrm{Ric}_{x_0}^{(M)}(u_i)$:= Ricci curvature at x_0 in direction u_i, $\Delta(u_0,\ldots,u_n)$:= $\mathrm{vol}^{(\mathbb{R}^n)}(\mathrm{Conv}(u_0,\ldots,u_n))$

- Volume of small balls

Theorem (Bertrand-Diguet-Puiseux)

$$\mathrm{vol}^{(M)}(\mathcal{B}^{(M)}(x_0,r))=\mathrm{vol}^{(\mathbb{R}^n)}(\mathcal{B}^{(\mathbb{R}^n)}(o,r))\left(1-\frac{\mathrm{Sc}_{x_0}^{(M)}}{6(n+2)}r^2+o(r^2)\right)$$

Sketch of proof: expansion of the Jacobian

- Each partial derivative is a Jacobi field J along a geodesic γ which satisfies both the Jacobi equation

$$J'' = \mathcal{R}_{\gamma(t)}(\gamma'(t), J(t))\gamma'(t)$$

and the Rauch comparison theorem

$$\|J(t)\|_{\gamma(t)} = t - \frac{K^{(M)}_{\gamma(0)}(J'(0), \gamma'(0))}{6} t^3 + o(t^3).$$

$\mathcal{R}_{\gamma(t)}(\cdot,\cdot) :=$ curvature tensor at $\gamma(t)$, $K^{(M)}_{\gamma(0)}(\cdot,\cdot) :=$ sectional curvature at $\gamma(0)$

- Expansion of each entry of the Jacobian matrix and careful calculation of the determinant

Further results

- Density of vertices in a fixed direction
- Same quantities for sectional tessellations
- Exact formulae in the case of the constant curvature

Plan

Mean asymptotics for the number of vertices of a Voronoi cell

Limit theorems for the empirical mean of the number of vertices

- Motivation
- Mean asymptotics
- Variance asymptotics
- Central limit theorem
- Estimation of $\mathrm{Sc}_{x_0}^{(M)}$

Motivation

Aim Use the knowledge of the Poisson-Voronoi graph to recover the curvature at x_0

- Replace $\mathbb{E}(\mathcal{N}(\mathcal{C}^{(M)}_{x_0,\lambda}))$ with an empirical mean around x_0

  $N^{(M)}_\lambda := \sum_{x \in \mathcal{B}(x_0, \lambda^{-\beta}) \cap \mathcal{P}_\lambda} \mathcal{N}(C^{(M)}(x, \mathcal{P}_\lambda)), \; {\scriptstyle 0 < \beta < 1/n}$

- Show expectation, variance asymptotics and a CLT for $N^{(M)}_\lambda$

- Deduce the construction of an estimator of $\mathrm{Sc}^{(M)}_{x_0}$ which is asymptotically unbiased, consistent and normal.

Mean asymptotics of $N_\lambda^{(M)}$

$$\mathbb{E}(N_\lambda^{(M)}) \underset{\lambda\to\infty}{=} \kappa_n \lambda^{1-n\beta}(e_n - d_n\,\mathrm{Sc}_{x_0}^{(M)}\,\lambda^{-\frac{2}{n}}) + o(\lambda^{1-n\beta-\frac{2}{n}})$$

where κ_n:= volume of the n-dimensional unit-ball

Sketch of proof

- $\xi(x,\mathcal{P}_\lambda) := \mathcal{N}(C^{(M)}(x,\mathcal{P}_\lambda))$

  $$\mathbb{E}(N_\lambda^{(M)}) = \lambda \int_{x\in\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})} \mathbb{E}(\xi(x,\mathcal{P}_\lambda))\mathrm{d}\mathrm{vol}^{(M)}(x)$$

- We require the uniformity of the two-term expansion of $\mathbb{E}(\xi(x,\mathcal{P}_\lambda))$ with respect to x.

Variance asymptotics for $N_\lambda^{(M)}$

$\frac{1}{3n} < \beta < \frac{1}{n}$

$$\mathrm{Var}(N_\lambda^{(M)}) \underset{\lambda\to\infty}{\sim} \kappa_n \lambda^{1-n\beta}\sigma_\infty$$

where κ_n:= volume of the n-dimensional unit-ball
and $\sigma_\infty := \mathbb{E}(\mathcal{N}(\mathcal{C}_{o,1}^{(\mathbb{R}^n)})^2) + \int \mathrm{Cov}(\mathcal{N}(\mathcal{C}^{(\mathbb{R}^n)}(x, \mathcal{P}_1 \cup \{x\})), \mathcal{N}(\mathcal{C}_{o,1}^{(\mathbb{R}^n)}))\mathrm{d}x$.

$\rightsquigarrow$ $(N_\lambda^{(M)} - \mathbb{E}(N_\lambda^{(M)}))$ of order $\sqrt{\mathrm{Var}(N_\lambda^{(M)})}$, i.e. $\lambda^{\frac{1}{2}(1-n\beta)}$
must be negligible in front of the second term of $\mathbb{E}(N_\lambda^{(M)})$, i.e. $\lambda^{1-n\beta-\frac{2}{n}}$.

Sketch of proof for the variance asymptotics

$$
\begin{aligned}
&\mathrm{Var}(N_\lambda^{(M)}) \\
&= \mathbb{E}\left(\sum_x \xi^2(x,\mathcal{P}_\lambda) + \sum_{x\neq y}\xi(x,\mathcal{P}_\lambda)\xi(y,\mathcal{P}_\lambda)\right) - \mathbb{E}(N_\lambda^{(M)})^2 \\
&= \lambda\int_{\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})}\mathbb{E}(\xi^2(x,\mathcal{P}_\lambda\cup\{x\}))\mathrm{dvol}^{(M)}(x) \\
&\quad + \lambda^2\iint_{\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})^2}\mathbb{E}(\xi(x,\mathcal{P}_\lambda\cup\{x,y\})\xi(y,\mathcal{P}_\lambda\cup\{x,y\}))\mathrm{dvol}^{(M)}(x)\mathrm{dvol}^{(M)}(y) \\
&\quad - \lambda^2\iint_{\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})^2}\mathbb{E}(\xi(x,\mathcal{P}_\lambda\cup\{x\}))\mathbb{E}(\xi(y,\mathcal{P}_\lambda\cup\{y\}))\mathrm{dvol}^{(M)}(x)\mathrm{dvol}^{(M)}(y) \\
&= \lambda\int_{\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})}\mathbb{E}(\xi^2(x,\mathcal{P}_\lambda\cup\{x\}))\mathrm{dvol}^{(M)}(x) \\
&\quad + \lambda^2\iint_{\mathcal{B}^{(M)}(x_0,\lambda^{-\beta})^2}\mathrm{Cov}(\xi(x,\mathcal{P}_\lambda\cup\{x\}),\xi(y,\mathcal{P}_\lambda\cup\{y\}))\mathrm{dvol}^{(M)}(x)\mathrm{dvol}^{(M)}(y)
\end{aligned}
$$

Sketch of proof for the variance asymptotics

Question Limits of $\mathbb{E}(\xi^2(x,\mathcal{P}_\lambda))$ and $\mathrm{Cov}(\xi(x,\mathcal{P}_\lambda),\xi(y,\mathcal{P}_\lambda))$?

- Application of the inverse of the exponential map at x_0, then a rescaling by $\lambda^{\frac{1}{n}}$ in the tangent space identified with $\mathbb{R}^n$
- Comparison of the obtained tessellation with an Euclidean Poisson-Voronoi tessellation and convergence of the scores
- Each score has a localization radius R, i.e.

$$\xi^{(\lambda)}(x,\mathcal{P}_\lambda)=\xi^{(\lambda)}(x,\mathcal{P}_\lambda\cap\mathcal{B}^{(M)}(x,R))$$

where $\mathbb{P}\{R>\lambda^{-\frac{1}{n}}t\}\le ce^{-\frac{t^n}{c}}$ uniformly in x and λ.

Central limit theorem for $N_\lambda^{(M)}$

$\frac{1}{3n} < \beta < \frac{1}{n}$

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(\frac{N_\lambda^{(M)} - \mathbb{E}(N_\lambda^{(M)})}{\sqrt{\mathrm{Var}(N_\lambda^{(M)})}} \le t\right) - \mathbb{P}(\mathcal{N}(0,1) \le t)\right| \le c\lambda^{\frac{1}{2}(1-n\beta)}.$$

Sketch of proof

- F functional of $\mathcal{P}_\lambda$ with $\mathbb{E}(F) = 0$ and $\mathrm{Var}(F) = 1$
  Estimate of $d_K(F, \mathcal{N}(0,1))$ involving integrals of moments of consecutive Malliavin derivatives of F where

$$D_x F = F(\mathcal{P}_\lambda \cup \{x\}) - F(\mathcal{P}_\lambda)$$

G. Last, G. Peccati & M. Schulte (2016)

- Uniform bounds for the moments of $D_x N_\lambda^{(M)}$ and $D^2_{x,y} N_\lambda^{(M)}$

Estimation of $\mathrm{Sc}_{x_0}^{(M)}$

$0 < \beta < \frac{1}{n} - \frac{4}{n^2}$ and $n \geq 6$

- $\widehat{\mathrm{Sc}}_\lambda(x_0) := \frac{\lambda^{\frac{2}{n}}}{d_n}\left(e_n - \frac{1}{\lambda \mathrm{vol}^{(M)}(B^{(M)}(x_0, \lambda^{-\beta}))} N_\lambda^{(M)}\right)$

is an asymptotically unbiased, consistent and normal estimator of $\mathrm{Sc}_{x_0}^{(M)}$.

- $\widehat{\widehat{\mathrm{Sc}}}_\lambda(x_0) := \frac{\lambda^{\frac{2}{n}}}{d_n}\left(e_n - \frac{1}{\#(\mathcal{P}_\lambda \cap B^{(M)}(x_0, \lambda^{-\beta}))} N_\lambda^{(M)}\right)$

is an asymptotically unbiased and consistent estimator of $\mathrm{Sc}_{x_0}^{(M)}$.

Thank you for your attention!