Stokes equation with Navier Boundary Condition and some limiting cases

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Outline

Introduction and motivation

First we consider in a bounded domain Ω in $\mathbb{R}^3$ with boundary Γ, possibly not connected, of class $\mathcal{C}^{1,1}$, the following Stokes equations

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = \mathbf{0} \quad \text{ in } \Omega$$

where the unknowns $\mathbf{u}$ and π stand respectively for the velocity field and the pressure of the fluid occupying a domain Ω. Given data is the external force $\mathbf{f}$. To study the Stokes equations it is necessary to add some suitable boundary conditions.

- Concerning these equations, the first thought goes to the classical no-slip Dirichlet boundary condition which is not always appropriate. For example it shows the absence of collisions of rigid bodies immersed in a linearly viscous fluid.
- In some applications, in particular in the electromagnetism problems, it is possible to find problems where it is necessary to consider other boundary conditions (BC). These BC are also used to simulate flows near rough walls, such as in aerodynamics, in weather forecasts and in hemodynamics, as well as perforated walls. BC involving the pressure, such as in cases of pipes, hydraulic gears using pomps, containers, etc ...

An alternative to the no-slip BC was suggested by H. Navier in 1823. Navier proposed a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangent to the surface should be proportional to the rate of strain at the surface

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad 2\left[\mathbb{D}(\mathbf{u})\mathbf{n}\right]_\tau + \alpha\, \mathbf{u}_\tau = \mathbf{h} \quad \text{on } \Gamma$$

where $\mathbb{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ denotes the deformation tensor associated to the velocity field $\mathbf{u}$ and α is the friction coefficient which is a scalar function.
Observe that if α tends to infinity, we get formally

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \qquad \Gamma.$$

The Navier boundary conditions are often used to simulate flows near rough walls as well as perforated walls.
Such slip boundary conditions are used in the Large Eddy Simulations (LES) of turbulent flows.

Our aim is to study the system

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = \mathbf{0} & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad 2\left[\mathbb{D}(\mathbf{u})\mathbf{n}\right]_\tau + \alpha\, \mathbf{u}_\tau = \mathbf{h} & \text{on } \Gamma \end{cases} \tag{S}$$

considering α not regular.

We first briefly review some existing or related works.

Literature

Stationary problem :

- Solonnikov-Scadilov, 1973, $\alpha = 0$ Hilbert case
- B.Da Veiga, 2004, $\alpha > 0$ constant, Hilbert case
- Berselli, 2010, $\alpha = 0$, flat domain in $\mathbb{R}^3$
- Amrouche-Rejaiba, 2014, $\alpha = 0$
- Verfurth, 1987.

Non-stationary problem :

- Mikelić et al, 1998, 2D, $\alpha \in C^2(\Gamma)$
- Kelliher, 2006, 2D, $\alpha \in L^\infty(\Gamma)$
- B.Da Veiga, 2007, 3D, $\alpha > 0$ constant
- Iftimie-Sueur, 2011, 3D, $\alpha \in C^2(\Gamma)$

Basic properties and useful inequalities

To study the problem, we consider the following assumptions on α:
$\alpha \in L^{t(p)}(\Gamma)$ with

$$t(p) = \begin{cases} \frac{2}{3}p' + \varepsilon & \text{if} \quad 1 < p < \frac{3}{2} \\ 2 + \varepsilon & \text{if} \quad \frac{3}{2} \leq p \leq 3, p \neq 2 \\ 2 & \text{if} \quad p = 2 \\ \frac{2}{3}\, p + \varepsilon & \text{if} \quad p > 3 \end{cases} \tag{0.1}$$

where $\varepsilon > 0$ is an arbitrary number, sufficiently small. Also, $\exists\ \alpha_*$ such that

$$\alpha \geq \alpha_* \geq 0 \tag{0.2}$$

with

$$\alpha_* \geq 0 \quad \text{if } \Omega \text{ is not axisymmetric} \tag{0.3a}$$

or

$$\alpha_* > 0 \quad \text{on } \Gamma_0 \subsetneq \Gamma \quad \text{if } \Omega \text{ is axisymmetric} \tag{0.3b}$$

or

$$\alpha_* = 0 \;\; \text{on } \Gamma \quad \text{if } \Omega \text{ is axisymmetric .} \tag{0.3c}$$

Note that the kernel of the system (S) corresponding to $\alpha = 0$ is:

$$\mathcal{T}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega) : \mathbb{D}(\mathbf{v}) = \mathbf{0} \text{ in } \Omega \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\right\}$$
$$= \begin{cases} \{\mathbf{0}\} & \text{if } \Omega \text{ is not axisymmetric} \\ \text{span}\{\mathbf{b}\times\mathbf{x}\} & \text{if } \Omega \text{ is axisymmetric} \end{cases}$$

But the kernel of the system (S) corresponding to $\alpha \neq 0$ is:

$$\mathcal{I}(\Omega) = \Big\{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega) : -\Delta\mathbf{u} + \nabla\pi = \mathbf{0},\ \text{div }\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n} = 0,$$
$$2\left[\mathbb{D}(\mathbf{u})\mathbf{n}\right]_\tau + \alpha\mathbf{u}_\tau = \mathbf{0} \text{ on } \Gamma\}$$
$$= \{\mathbf{0}\}.$$

Let us first discuss some Korn-type inequalities which will be used to prove the equivalence of norms and the existence of solution.

Proposition

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, Lipschitz. For all $\mathbf{u} \in \mathbf{H}^1(\Omega)$ with $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ, the following equivalence of norms hold:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \simeq \|\mathbb{D}(\mathbf{u})\|_{\mathbf{L}^2(\Omega)} \quad \textit{if } \Omega \textit{ is not axisymmetric },$$

and

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \simeq \|\mathbb{D}(\mathbf{u})\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_\tau\|_{\mathbf{L}^2(\Gamma_0)} \quad \textit{if } \Omega \textit{ is axisymmetric }.$$

We also deduce the following inequalities:

Proposition

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. If Ω is axisymmetric with respect to a constant vector $\mathbf{b} \in \mathbb{R}^3$ and $\beta(\mathbf{x}) = \mathbf{b} \times \mathbf{x}$ for $\mathbf{x} \in \Omega$, then we have the following inequalities: for all $\mathbf{u} \in \mathbf{H}^1(\Omega)$ with $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ,

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \left[\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} + \left(\int_\Omega \mathbf{u} \cdot \beta \, \mathrm{d}x \right)^2 \right]$$

and

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \left[\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} + \left(\int_\Gamma \mathbf{u} \cdot \beta \, \mathrm{d}s \right)^2 \right].$$

These results can be proved by the method of contradiction.

Now consider $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ s.t. $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ where

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if} \quad p > \frac{3}{2} \\ 1 & \text{if} \quad 1 < p \leq \frac{3}{2}. \end{cases}$$

We call $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ is a weak solution of the problem (S) iff for all $\varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2 \int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi) \, \mathrm{d}x + \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau \, \mathrm{d}s = \int_\Omega \mathbf{f} \cdot \varphi \, \mathrm{d}x + \langle \mathbf{h}, \varphi \rangle_\Gamma. \quad (0.4)$$

It is easy to see from the above weak formulation that if $\alpha = 0$ and Ω is axisymmetric,

$$\int_\Omega \mathbf{f} \cdot \beta \, \mathrm{d}x + \langle \mathbf{h}, \beta \rangle_\Gamma = 0$$

is a necessary condition for the existence of a solution.

Note that the boundary term in the left hand side of the weak formulation (0.4) is actually a well-defined integral which can be seen from the following argument.

Since $\varphi \in \mathbf{W}^{1,p'}(\Omega)$, we have $\varphi_\tau \in \mathbf{W}^{1-\frac{1}{p'},p'}(\Gamma) \hookrightarrow \mathbf{L}^m(\Gamma)$ where

$$\frac{1}{m} = \begin{cases} 1 - \frac{3}{2p} & \text{if } p > \frac{3}{2}, \\ \text{any positive real number} < 1 & \text{if } p = \frac{3}{2}, \\ 0 & \text{if } p < \frac{3}{2}. \end{cases}$$

Similarly, for $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$, $\mathbf{u}_\tau \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \mathbf{L}^s(\Gamma)$ with

$$\frac{1}{s} = \begin{cases} \frac{3}{2p} - \frac{1}{2} & \text{if } p < 3, \\ \text{any positive real number} < 1 & \text{if } p = 3, \\ 0 & \text{if } p > 3. \end{cases}$$

Thus for $\alpha \in L^{t(p)}(\Gamma)$, it can be easily seen that $\alpha \mathbf{u}_\tau \in \mathbf{L}^{m'}(\Gamma)$.

L^2-Theory

The first theorem gives us the existence, uniqueness and estimates of the solution of (S).

Theorem

Let $\mathbf{f} \in \mathbf{L}^{\frac{6}{5}}(\Omega), \mathbf{h} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ *and* $\alpha \in L^2(\Gamma)$ *satisfying* (0.2)-(0.3). *Then the Stokes problem* (S) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ *with the following estimates,*

(I) if Ω *is not axisymmetric, then*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

(II) if Ω *is axisymmetric and*

- $\alpha \geq \alpha_* > 0$ *on* $\Gamma_0 \subseteq \Gamma$, *then*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

- $\mathbf{f}, \mathbf{h}$ satisfy the condition:

$$\int_\Omega \mathbf{f} \cdot \beta \, \mathrm{d}x + \langle \mathbf{h}, \beta \rangle_\Gamma = 0$$

then, the solution $\mathbf{u}$ satisfies $\int_\Gamma \alpha \mathbf{u} \cdot \beta \, \mathrm{d}s = 0$ and

$$\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} + \int_\Gamma \alpha |\mathbf{u}_\tau|^2 \, \mathrm{d}s \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)^2 .$$

Moreover, if α is a constant, then

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right) .$$

Proof. The existence and uniqueness follows from the Lax-Milgram Lemma (where the coercivity of the bilinear form is obvious) and also the estimate. But note that the continuity constant we get from Lax-Milgram Lemma depends on α. So we prove independently the different estimates, independent of α for that we use the previously stated Korn-type inequalities and equivalence of norms.

Next we prove the existence of strong solution and the corresponding estimate independent of α.

Theorem

Let Ω be $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{h} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ and α is a constant, satisfying (0.2)-(0.3). Then the solution of (S) belongs to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$, satisfying the following estimate,

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right). \qquad (0.5)$$

Remark. Later we will prove the existence result of strong solution for more general α, not necessarily a constant.

Proof.

Method I : If α is a constant and $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{h} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, then $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and therefore $\alpha \mathbf{u}_\tau \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ also. So using the regularity result as in the paper Amrouche-Rejaiba, we get that $\mathbf{u} \in \mathbf{H}^2(\Omega)$.

But concerning the estimate, with this method, we can not obtain the bound on $\mathbf{u}$, independent of α. Thus we need to consider the more fundamental but long method, explained below.

Method II : The proof is based on the method of difference quotient as in the book of L.C. Evans. Without loss of generality, we consider $\mathbf{h} = \mathbf{0}$. Also, let denote the difference quotient by,

$$D_k^h \mathbf{u}(x) = \frac{\mathbf{u}(x + h\mathbf{e}_k) - \mathbf{u}(x)}{h}, \quad k = 1, 2, 3, \quad h \in \mathbb{R}^\cdot$$

1. Interior regularity. Showing that $(\mathbf{u}, \pi)$ belongs to $\mathbf{H}^2_{loc}(\Omega) \times H^1_{loc}(\Omega)$ with the estimate (0.5), is very classical, with the help of $\mathbf{H}^1$-estimate, since the method does not depend on the boundary condition.

2. Boundary regularity.

   The solution $(\mathbf{u}, \pi)$ satisfies the variational formulation, for all $\varphi \in \mathbf{H}^1_\tau(\Omega)$,

$$2\int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi)\, \mathrm{d}x + \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau\, \mathrm{d}s - \int_\Omega \pi\, \mathrm{div}\varphi\, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \varphi\, \mathrm{d}x. \tag{0.6}$$

   Case 1 : Ω is a half-ball i.e. $\Omega = B(0,1) \cap \mathbb{R}^3_+$.
   Set $V := B(0, \frac{1}{2}) \cap \mathbb{R}^3_+$. Then choose a cut-off function $\zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{cases} \zeta \equiv 1 \text{ on } B(0, \frac{1}{2}),\ \zeta \equiv 0 \textbf{ on } \mathbb{R}^3 \setminus B(0,1), \\ 0 \leq \zeta \leq 1. \end{cases}$$

   So $\zeta \equiv 1$ on V and vanishes on the curved part of Γ.

- **tangential regularity of u :**

Let $h > 0$ be small and $\varphi = -D_k^{-h}(\zeta^2 D_k^h \mathbf{u}), k = 1, 2$. Clearly, $\varphi \in \mathbf{H}^1_\tau(\Omega)$. Therefore, we can substitute φ into the identity (0.6) and obtain,

$$\begin{aligned} &2\int_\Omega \zeta^2 |D_k^h \mathbb{D}(\mathbf{u})|^2 \,\mathrm{d}x + 2\int_\Omega D_k^h \mathbb{D}(\mathbf{u}) : 2\zeta \nabla\zeta D_k^h \mathbf{u} \,\mathrm{d}x \\ &+ \int_\Gamma \alpha \zeta^2 |D_k^h \mathbf{u}_\tau|^2 \,\mathrm{d}s - \int_\Omega \pi \,\mathrm{div}(-D_k^{-h}(\zeta^2 D_k^h \mathbf{u})) \,\mathrm{d}x \\ &= \int_\Omega \mathbf{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \mathbf{u})) \,\mathrm{d}x. \end{aligned} \tag{0.7}$$

Now we estimate the different terms. For the second term in the left hand side, using Cauchy's inequality with ϵ, we get

$$\begin{aligned} &|\int_\Omega D_k^h \mathbb{D}(\mathbf{u}) : 2\zeta \nabla\zeta D_k^h \mathbf{u} \,\mathrm{d}x| \\ &\leq C \int_\Omega 2\zeta |D_k^h \mathbb{D}(\mathbf{u})||D_k^h \mathbf{u}| \,\mathrm{d}x \\ &\leq C \left[\epsilon \int_\Omega \zeta^2 |D_k^h \mathbb{D}(\mathbf{u})|^2 \,\mathrm{d}x + \frac{1}{\epsilon} \int_\Omega |D_k^h \mathbf{u}|^2 \,\mathrm{d}x \right]. \end{aligned}$$

Similarly we can estimate the fourth term in the left hand side and the term in the right hand side which gives us altogether

$$2\int_\Omega \zeta^2 |D_k^h \mathbb{D}(\mathbf{u}|)^2 \, \mathrm{d}x + \int_\Gamma \alpha \, \zeta^2 |D_k^h \mathbf{u}_\tau|^2 \, \mathrm{d}s$$
$$\leq \epsilon \int_\Omega \zeta^2 |\nabla D_k^h \mathbf{u}|^2 \, \mathrm{d}x + \frac{C_1}{\epsilon} \left(\int_\Omega |\mathbf{f}|^2 \, \mathrm{d}x + \int_\Omega |\pi|^2 \, \mathrm{d}x \right) + C_2 \int_\Omega |D_k^h \mathbf{u}|^2 \, \mathrm{d}x.$$

From here, we deduce

$$\|\zeta D_k^h \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \leq \epsilon \|\zeta D_k^h \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \frac{C_1}{\epsilon} \left(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\pi\|^2_{\mathbf{L}^2(\Omega)} \right) + C_2 \|D_k^h \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$$

Choosing ϵ small and using the estimates for $(\mathbf{u}, \pi)$ in $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$, we obtain,

$$\|D_k^h \mathbf{u}\|^2_{\mathbf{H}^1(V)} \leq \|\zeta D_k^h \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \leq C(\Omega) \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}$$

which means that $\partial^2 \mathbf{u} / \partial x_i \partial x_j$ belongs to $\mathbf{L}^2(V)$ for all $i, j = 1, 2, 3$ except for $i = j = 3$, with the corresponding estimate.

- **tangential regularity of π :** From the Stokes equation, for $i = 1, 2$, we get,

$$\frac{\partial}{\partial x_i}(\nabla \pi) = \frac{\partial}{\partial x_i}(\mathbf{f} + \Delta \mathbf{u}) = \frac{\partial \mathbf{f}}{\partial x_i} + \operatorname{div}(\nabla \frac{\partial \mathbf{u}}{\partial x_i}).$$

Since there is no term of the form $\partial^2 \mathbf{u}/\partial x_3^2$, by preceding arguments, we obtain

$$\nabla \frac{\partial \pi}{\partial x_i} = \frac{\partial}{\partial x_i}(\nabla \pi) \in \mathbf{H}^{-1}(V).$$

Furthermore, as we already know that $\frac{\partial \pi}{\partial x_i} \in H^{-1}(V)$, hence from the Nečas inequality, $\frac{\partial \pi}{\partial x_i} \in L^2(V)$ and satisfies the usual estimate.

- **normal regularity :** For the complete regularity of the solution, it remains to study the derivatives of $\mathbf{u}$ and π in the direction $\mathbf{e}_3$. Differentiating the divergence equation with respect to x_3 gives,

$$\frac{\partial^2 u_3}{\partial x_3^2} = -\sum_{i=1}^{2} \frac{\partial^2 u_i}{\partial x_i \partial x_3} \in L^2(V).$$

Next, from the 3rd component of the Stokes equation, we can write,

$$\frac{\partial \pi}{\partial x_3} = f_3 + \Delta u_3 \in L^2(V)$$

which proves that $\pi \in H^1(V)$. Finally, for $i = 1, 2$, we can write the ith equation of the system in the form,

$$\frac{\partial^2 u_i}{\partial x_3^2} = -\sum_{j=1}^{2} \frac{\partial^2 u_i}{\partial x_j^2} - f_i + \frac{\partial \pi}{\partial x_i} \in L^2(V).$$

Thus, $u_i \in H^2(V)$. So, apart from the regularity of $\mathbf{u}$ and π, we have proved the existence of $C = C(\Omega) > 0$ independent of α such that

$$\|\mathbf{u}\|_{\mathbf{H}^2(V)} + \|\pi\|_{H^1(V)} \leq C\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Case 2 : Now we drop the assumption that Ω is a half ball and consider the general case. Since Γ is $\mathcal{C}^{2,1}$, for any $x_0 \in \Gamma$, we can assume, upon relabelling the coordinate axes,

$$\Omega \cap (B(x_0, r)) = \{x \in (B(x_0, r)) : x_3 > H(x')\}$$

for some $r > 0$ and $H : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{2,1}$.

Let us now introduce the change of variable,

$$y = (x_1, x_2, x_3 - H(x')) := \phi(x)$$

i.e.

$$x = (y_1, y_2, y_3 + H(y')) := \phi^{-1}(y).$$

Choose $s > 0$ small so that the half ball $\Omega' := B(0, s) \cap \mathbb{R}^3_+$ lies in $\phi(\Omega \cap (B(x_0, r)))$. Set also $V' := B(0, s/2) \cap \mathbb{R}^3_+$ and $\mathbf{u}'(y) = \mathbf{u}(\phi^{-1}(y))$ for $y \in \Omega'$. It is easy to see that

$$\mathbf{u}' \in \mathbf{H}^1(\Omega')$$

and

$$\mathbf{u}' \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega' \cap \partial\mathbb{R}^3_+$$

but

$$\text{div } \mathbf{u} = \text{div } \mathbf{u}' - \sum_{j=1}^{2} \frac{\partial H}{\partial y_j} \frac{\partial u'_j}{\partial y_3}.$$

Now transforming our problem to the new coordinates, under this change of variable, (0.6) becomes,

$$
\begin{aligned}
&2\int_{\Omega'} \mathbb{D}(\mathbf{u}') : \mathbb{D}(\varphi')\, \mathrm{d}y + \int_{\Gamma'} \alpha \mathbf{u}'_\tau \cdot \varphi'_\tau\, \mathrm{d}s - \int_{\Omega'} \pi' \,\mathrm{div}\varphi'\, \mathrm{d}y \\
&= \int_{\Omega'} \mathbf{f}' \cdot \varphi'\, \mathrm{d}y - \int_{\Omega'} \pi' \frac{\partial H}{\partial y_j} \frac{\partial u'_j}{\partial y_3}\, \mathrm{d}y + \int_{\Omega'} \partial H\; \nabla \mathbf{u}' \nabla \varphi'\, \mathrm{d}y.
\end{aligned} \tag{0.8}
$$

We choose the test function $\varphi' = -D_k^{-h}(\zeta^2 D_k^h \mathbf{u}')$ as before and estimate the extra terms. For the second term in the right hand side, it is easy to see,

$$
\left| \int_{\Omega'} \pi' \frac{\partial H}{\partial y_j} \frac{\partial u'_j}{\partial y_3}\, \mathrm{d}y \right| \leq C \left(\|\pi'\|^2_{L^2(\Omega')} + \|\nabla \mathbf{u}'\|^2_{\mathbf{L}^2(\Omega')} \right).
$$

And for the last term in the right hand side, we get

$$
\begin{aligned}
&\left| \int_{\Omega'} \partial H\; \nabla \mathbf{u}' \nabla (D_k^{-h}(\zeta^2 D_k^h \mathbf{u}'))\, \mathrm{d}y \right| \\
\leq &C \left(\int_{\Omega'} |\nabla \mathbf{u}'|^2\, \mathrm{d}y + \epsilon \int_{\Omega'} \zeta^2 |\nabla D_k^h \mathbf{u}'|^2\, \mathrm{d}y + \frac{1}{\epsilon} \int_{\Omega'} |D_k^h \mathbf{u}'|^2\, \mathrm{d}y \right. \\
&\qquad \left. + \int_{\Omega'} \zeta^2 |D_k^h \nabla \mathbf{u}'|^2 |\partial H|\, \mathrm{d}y \right).
\end{aligned}
$$

Hence, accumulating all these inequalities, we obtain from (0.8),

$$\|\zeta D_k^h \mathbf{u}'\|_{\mathbf{H}^1(\Omega')}^2 \leq C \Big(\|\nabla \mathbf{u}'\|_{\mathbf{L}^2(\Omega')}^2 + \|\pi'\|_{L^2(\Omega')}^2 + \|\mathbf{f}'\|_{\mathbf{L}^2(\Omega')}^2 \\ + \epsilon \|\zeta D_k^h \mathbf{u}'\|_{\mathbf{H}^1(\Omega')}^2 + \int_{\Omega'} \zeta^2 |\partial H| |\nabla D_k^h \mathbf{u}'|^2 \, \mathrm{d}y \Big) .$$

But $|\partial H|$ is small for sufficiently small $s > 0$, since

$$\frac{\partial H}{\partial y_1}(0,0) = 0 = \frac{\partial H}{\partial y_2}(0,0)$$

which gives,

$$\|D_k^h \mathbf{u}'\|_{\mathbf{H}^1(V')}^2 \leq \|\zeta D_k^h \mathbf{u}'\|_{\mathbf{H}^1(\Omega')}^2 \leq C \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2. \qquad (0.9)$$

Then, proceeding as in the case of the half ball, we can deduce,

$$\mathbf{u}' \in \mathbf{H}^2(V') \quad \text{and} \quad \|\mathbf{u}'\|_{\mathbf{H}^2(V')} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Consequently,

$$\|\mathbf{u}\|_{\mathbf{H}^2(V)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Since, Γ is compact, we can cover Γ with finitely many sets $\{V_i\}$ as above. Summing the resulting estimates, along with the interior estimate, we get $\mathbf{u} \in \mathbf{H}^2(\Omega)$ with the desired inequality.

L^p-Theory

We begin with recalling some important results.

Theorem (Inf-Sup condition in Banach spaces)

Let X and M be two reflexive Banach spaces and X' and M' be their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the operators defined by

$$\forall v \in X, \quad \forall w \in M, \quad a(v, w) = \langle Av, w \rangle = \langle v, A'w \rangle$$

and $V = Ker\ A$. Then the following statements are equivalent :
(i) There exists $C = C(\Omega) > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \ \sup_{\substack{v \in X \\ v \neq 0}} \ \frac{a(v, w)}{\|v\|_X \ \|w\|_M} \geq C \ . \tag{0.10}$$

(ii) The operator $A : X/V \mapsto M'$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of A^{-1}.
(iii) The operator $A' : M \mapsto X' \perp V$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of $(A')^{-1}$.

Also, recall the following Inf-Sup condition (Amrouche-Seloula [Lemma 4.4]):

Lemma

There exists a constant $\beta > 0$ such that

$$\inf_{\substack{\varphi \in \mathbf{V}^{p'}(\Omega) \\ \varphi \neq \mathbf{0}}} \sup_{\substack{\xi \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega) \\ \xi \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \xi \cdot \mathbf{curl}\, \varphi \,\mathrm{d}x}{\|\xi\|_{\mathbf{V}^{p}_{\sigma,\tau}(\Omega)} \|\varphi\|_{\mathbf{V}^{p'}(\Omega)}} \geq C \tag{0.11}$$

where

$$\mathbf{V}^{p'}(\Omega) := \left\{ \mathbf{v} \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega);\ \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0 \quad \forall\, 1 \leq j \leq J \right\} .$$

Next we consider the bilinear form: for $\mathbf{u} \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega)$ and $\varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$a(\mathbf{u}, \varphi) = 2 \int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi) \,\mathrm{d}x + \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau \,\mathrm{d}s$$

and prove a more general result.

Theorem

Let Ω *be* $\mathcal{C}^{2,1}$, $1 < p < \infty$ *and*

$$\ell \in [\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)]', \ \alpha \in L^{t(p)}(\Gamma) \ \textit{satisfying} \ (0.2)\text{-}(0.3)$$

where in addition, we suppose the following compatibility condition when $\alpha = 0$ *and* Ω *is axisymmetric,*

$$\langle \ell, \beta \rangle = 0.$$

Then the problem:

$$\textit{Find } \mathbf{u} \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega) \textit{ s.t. for any } \varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega), \ a(\mathbf{u}, \varphi) = \langle \ell, \varphi \rangle \quad (0.12)$$

has a unique solution.

Proof. Observe first that if Ω is axisymmetric and $\alpha = 0$, from the formulation of problem (0.12), we can see immediately the necessity of the compatibility condition.

- $p \geq 2$.

Since $[\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)]' \hookrightarrow [\mathbf{V}^{2}_{\sigma,\tau}(\Omega)]'$, we deduce by Lax-Milgram lemma that there exists a unique $\mathbf{u} \in \mathbf{V}^{2}_{\sigma,\tau}(\Omega)$ satisfying :

$$\forall\, \varphi \in \mathbf{V}^{2}_{\sigma,\tau}(\Omega), \quad a(\mathbf{u}, \varphi) = \langle \ell, \varphi \rangle_{[\mathbf{V}^{2}_{\sigma,\tau}(\Omega)]' \times \mathbf{V}^{2}_{\sigma,\tau}(\Omega)} \,. \tag{0.13}$$

Now we will prove that $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$. Since the Inf-Sup condition (0.10) is known for the bilinear form

$$b(\mathbf{u}, \varphi) = \int_{\Omega} \mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \varphi \,\mathrm{d}x$$

with adapted spaces X and M and we have the relation

$$[2\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{u} \quad \text{on}\, \Gamma,$$

where Λ is an operator of order 0 defined by

$$\boldsymbol{\Lambda}\boldsymbol{u} = \sum_{k=1}^{2} \big(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\big)\boldsymbol{\tau}_k,$$

we will use another formulation of problem (0.13).

Using the density result,

$$\left\{\mathbf{v} \in \mathbf{H}^2(\Omega),\ \text{div}\ \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\} \text{ is dense in } \mathbf{V}^2_{\sigma,\tau}(\Omega)$$

we get from (0.13), for all $\varphi \in \mathbf{V}^2_{\sigma,\tau}(\Omega)$,

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\varphi\, \mathrm{d}x = \langle \ell, \varphi \rangle_{[\mathbf{V}^2_{\sigma,\tau}(\Omega)]' \times \mathbf{V}^2_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau\, \mathrm{d}s + 2\int_\Gamma \mathbf{\Lambda u} \cdot \varphi\, \mathrm{d}s. \tag{0.14}$$

Now we are in position to improve the regularity of $\mathbf{u}$ and for that we use bootstrap argument.

Case (I) : $2 < p \leq 3.$

Step 1. Since $\mathbf{u}_\tau \in \mathbf{L}^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha \mathbf{u}_\tau \in \mathbf{L}^{q_1}(\Gamma)$ where $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But, $\mathbf{L}^{q_1}(\Gamma) \hookrightarrow \mathbf{W}^{-\frac{1}{p_1}, p_1}(\Gamma)$ with $p_1 = \frac{3}{2} q_1 \ > 2$ i.e. $\frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right)$. Therefore, as $\mathbf{W}^{\frac{1}{p_1}, p_1'}(\Gamma) \hookrightarrow \mathbf{L}^{q_1'}(\Gamma)$ with $\frac{4}{3} < q_1' < 4$ and $\mathbf{\Lambda u} \in \mathbf{L}^4(\Gamma)$, the mapping

$$\langle \mathbf{L}, \varphi \rangle = \langle \ell, \varphi \rangle - \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau\, \mathrm{d}s + 2\int_\Gamma \mathbf{\Lambda u} \cdot \varphi\, \mathrm{d}s \quad \text{for} \ \ \varphi \in \mathbf{V}^{s_1'}(\Omega)$$

defines an element of the dual space of $\mathbf{V}^{s_1'}(\Omega)$ with $s_1 = \min\{p_1, p\}$.

Now from the Inf-Sup condition (0.11), $\exists$ a unique $\mathbf{v} \in \mathbf{V}^{s_1}_{\sigma,\tau}(\Omega)$ s.t.

$$\forall \varphi \in \mathbf{V}^{s_1'}(\Omega), \quad \int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{curl}\ \varphi \, \mathrm{d}x = \langle \mathbf{L}, \varphi \rangle_{[\mathbf{V}^{s_1'}(\Omega)]' \times \mathbf{V}^{s_1'}(\Omega)} . \quad (0.15)$$

We will show that $\mathbf{curl}\ \mathbf{v} = \mathbf{curl}\ \mathbf{u}$. For that first we extend (0.15) to any test function $\varphi \in \mathbf{V}^{s_1'}_{\sigma,\tau}(\Omega)$ and since $\mathbf{V}^{2}_{\sigma,\tau}(\Omega) \hookrightarrow \mathbf{V}^{s_1'}_{\sigma,\tau}(\Omega)$, we deduce from (0.14) that

$$\forall \varphi \in \mathbf{V}^{2}_{\sigma,\tau}(\Omega), \quad \int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{curl}\ \varphi \, \mathrm{d}x = \int_\Omega \mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \varphi \, \mathrm{d}x$$

which gives,

$$\mathbf{curl}\ \mathbf{u} = \mathbf{curl}\ \mathbf{v} \quad \text{in} \ \ \Omega .$$

Therefore, as $\mathbf{u} \in \mathbf{L}^6(\Omega) \hookrightarrow \mathbf{L}^{s_1}(\Omega), \mathbf{curl}\ \mathbf{u} \in \mathbf{L}^{s_1}(\Omega), \mathrm{div}\ \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ; we get $\mathbf{u} \in \mathbf{W}^{1,s_1}(\Omega)$. If $s_1 \geq p$, the proof is complete. Otherwise, $s_1 = p_1$ and we proceed to the next step.

Step 2. Now, $\mathbf{u} \in \mathbf{W}^{1,p_1}(\Omega)$ implies the mapping

$$\langle \mathbf{L}, \varphi \rangle = \langle \ell, \varphi \rangle - \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau \, \mathrm{d}s + 2 \int_\Gamma \mathbf{\Lambda} \mathbf{u} \cdot \varphi \, \mathrm{d}s \quad \text{for} \ \ \varphi \in \mathbf{V}^{s_2'}(\Omega)$$

defines an element of the dual space of $\mathbf{V}^{s_2'}(\Omega)$ with $s_2 = \min\{p_2, p\}$ where $\frac{1}{p_2} = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right)$. Therefore, as in the previous step, $\exists$ a unique $\mathbf{v} \in \mathbf{V}^{s_2}_{\sigma,\tau}(\Omega)$ s.t.

$$\forall\ \varphi \in \mathbf{V}^{s_2'}_{\sigma,\tau}(\Omega), \qquad \int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{curl}\ \varphi\, \mathrm{d}x = \langle \mathbf{L}, \varphi \rangle$$

which implies again

$$\mathbf{curl}\ \mathbf{u} = \mathbf{curl}\ \mathbf{v} \quad \text{ in } \Omega.$$

Therefore, we get, $\mathbf{u} \in \mathbf{L}^{p_1^*}(\Omega) \hookrightarrow \mathbf{L}^{s_2}(\Omega)$, $\mathbf{curl}\ \mathbf{u} \in \mathbf{L}^{s_2}(\Omega)$, $\operatorname{div}\ \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ; which implies $\mathbf{u} \in \mathbf{W}^{1,s_2}(\Omega)$. If $s_2 \geq p$, we are done. Otherwise, $s_2 = p_2$ and we proceed next.

Step (k+1). Proceeding similarly, we get $\mathbf{u} \in \mathbf{V}^{p_{k+1}}_{\sigma,\tau}(\Omega)$ with $\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right)$ (where in each step, we assumed $p_k < 3$) which also satisfies, for all $\varphi \in \mathbf{V}^{p'_{k+1}}_{\sigma,\tau}(\Omega)$,

$$\int_\Omega \mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \varphi\, \mathrm{d}x = \langle \ell, \varphi \rangle - \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau\, \mathrm{d}s + 2\int_\Gamma \mathbf{\Lambda u} \cdot \varphi\, \mathrm{d}s.$$

Now choose $k = [\frac{1}{\varepsilon} - \frac{1}{2}] + 1$ such that $p_{k+1} \geq 3 \geq p$ (where $[a]$ stands for the greatest integer less than or equal to a). Hence $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$.

Case (II) : $p > 3$.

From the previous case, we get that $\mathbf{u} \in \mathbf{W}^{1,3}(\Omega)$ which implies $\mathbf{u}_\tau \in \mathbf{L}^s(\Gamma)$ for all $1 < s < \infty$. Then by similar argument (and in one iteration) we can deduce $\mathbf{u} \in \mathbf{V}^p_{\sigma,\tau}(\Omega)$ which solves the problem (0.12).

- p<2.

Consider the operator $A \in \mathcal{L}(\mathbf{V}^p_{\sigma,\tau}(\Omega), (\mathbf{V}^{p'}_{\sigma,\tau}(\Omega))')$, associated to the bilinear form a, defined as, $\langle A\xi, \varphi\rangle = a(\xi, \varphi)$. As described above, for $p \geq 2$, the operator A is an isomorphism from $\mathbf{V}^p_{\sigma,\tau}(\Omega)$ to $(\mathbf{V}^{p'}_{\sigma,\tau}(\Omega))'$. Then the adjoint operator, which is equal to A is an isomorphism from $\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)$ to $(\mathbf{V}^p_{\sigma,\tau}(\Omega))'$ for $p' \leq 2$. This means that the operator A is an isomorphism for $p \leq 2$ also, which ends the proof.

In particular, choosing $\langle \ell, \varphi \rangle = \int_\Omega \mathbf{f} \cdot \varphi \, \mathrm{d}x + \langle \mathbf{h}, \varphi \rangle_\Gamma$ in the above theorem, we get the following existence result.

Theorem

Let Ω *be* $\mathcal{C}^{2,1}$, $1 < p < \infty$ *and* $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$, $\alpha \in L^{t(p)}(\Gamma)$ *satisfying* (0.2)-(0.3). *Then the Stokes problem* (S) *has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.

Also, the general theorem yields the following interesting Inf-Sup condition.

Inf-Sup condition : for $1 < p < \infty$, $\exists \gamma = \gamma(\Omega, p, \alpha) > 0$ such that

$$\inf_{\substack{\varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega) \\ \varphi \neq 0}} \sup_{\substack{\xi \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega) \\ \xi \neq 0}} \frac{2 \int_\Omega \mathbb{D}(\xi) : \mathbb{D}(\varphi) \, \mathrm{d}x + \int_\Gamma \alpha \xi_\tau \cdot \varphi_\tau \, \mathrm{d}s}{\|\xi\|_{\mathbf{V}^{p}_{\sigma,\tau}(\Omega)} \, \|\varphi\|_{\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq \gamma \quad (0.16)$$

when Ω is not axisymmetric or $\alpha \neq 0$.

Question: Is it possible to obtain γ in the above inf-sup condition independent of α ?

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Next we discuss the L^p estimates of the solution.

Estimate

Let $p > 2$. The solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ of problem (S) satisfies the estimates:

(I) if Ω is not axisymmetric, then

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega) \left(1 + \|\alpha\|^2_{L^{t(p)}(\Gamma)}\right) \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{h}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

(II) if Ω is axisymmetric and

- $\alpha \geq \alpha_* > 0$ on $\Gamma_0 \subseteq \Gamma$, then

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega) \left(1 + \|\alpha\|^2_{L^{t(p)}(\Gamma)}\right) \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{h}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

- $\alpha \geq 0$ and $\mathbf{f}, \mathbf{h}$ satisfy the condition:

$$\int_\Omega \mathbf{f} \cdot \beta \, \mathrm{d}x + \langle \mathbf{h}, \beta \rangle_\Gamma = 0$$

then,

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)} \leq C(\Omega)\left(1+\|\alpha\|^2_{L^{t(p)}(\Gamma)}\right)\left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)}+\|\mathbf{h}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

Idea of the proof : To derive the estimates, we choose some suitable compactness argument. As it is similar to the other cases and follows from different $\mathbf{H}^1$-estimate, we sketch the proof of the first estimate.

Consider that Ω is not axisymmetric. Without loss of generality, we also assume $\mathbf{h} = \mathbf{0}$.

Case 2<p<3 : We know $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$. That means $\mathbf{u}_\tau \in \mathbf{L}^s(\Gamma)$ where $\frac{1}{s} = \frac{3}{2p} - \frac{1}{2}$. Also $\alpha \in L^{2+\epsilon}(\Gamma)$. Hence, $\alpha\mathbf{u}_\tau \in \mathbf{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{1}{s} + \frac{1}{2+\epsilon} < \frac{3}{2p}$. But as $\mathbf{L}^q(\Gamma) \hookrightarrow \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$, we get $\alpha\mathbf{u}_\tau \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$.
Now from the relation,

$$\mathbf{L}^q(\Gamma) \underset{\text{compact}}{\hookrightarrow} \mathbf{W}^{-\frac{1}{p},p}(\Gamma) \underset{\text{continuous}}{\hookrightarrow} \mathbf{H}^{-\frac{1}{2}}(\Gamma)$$

we can write, for any $\delta > 0$, $\exists$ a constant $C = C(p, \varepsilon, \Omega)$ (independent of δ) such that

$$\|\mathbf{v}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} \leq \delta \|\mathbf{v}\|_{\mathbf{L}^q(\Gamma)} + \frac{C}{\delta} \|\mathbf{v}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \qquad \forall \mathbf{v} \in \mathbf{L}^q(\Gamma) .$$

Choosing $\mathbf{v} = \alpha \mathbf{u}_\tau$ we get, using Hölder inequality and trace theorem,

$$\begin{aligned} \|\alpha\mathbf{u}_\tau\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} &\leq \delta \|\alpha\mathbf{u}_\tau\|_{\mathbf{L}^q(\Omega)} + \frac{C}{\delta} \|\alpha\mathbf{u}_\tau\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\leq C_1\delta \|\alpha\|_{L^{2+\epsilon}(\Gamma)}\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + C_2\frac{C}{\delta} \|\alpha\|_{L^2(\Gamma)}\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}. \end{aligned}$$

Now, using the regularity result from Amrouche-Rejaiba, we can get that

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \\ &\leq C\left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\alpha\mathbf{u}_\tau\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\leq C \|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + C_1\delta\|\alpha\|_{L^{2+\epsilon}(\Gamma)}\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \frac{C_2}{\delta}\|\alpha\|_{L^2(\Gamma)}\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}. \end{aligned}$$

Choose $\delta > 0$ small enough such that $1 - \delta C_1 \|\alpha\|_{L^{2+\epsilon}(\Gamma)} = \frac{1}{2}$ i.e. $\delta = \frac{1}{2C_1\|\alpha\|_{L^{2+\epsilon}(\Gamma)}}$. Thus we get,

$$
\begin{aligned}
& \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \\
\leq\ & 2C_1\ \|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + C_2\ \|\alpha\|_{L^2(\Gamma)} \|\alpha\|_{L^{2+\epsilon}(\Gamma)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \\
\leq\ & C(1 + \|\alpha\|^2_{L^{(2+\epsilon)}(\Gamma)}) \|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)}.
\end{aligned}
$$

Case $p \geq 3$: The analysis is exactly similar to the previous case, just based on different embedding results.

Theorem (Strong solution)

Ω is $\mathcal{C}^{2,1}$ and $1 < p < \infty$. $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{h} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma)$ with $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ and

$$
\alpha \in \begin{cases} W^{1-\frac{1}{\frac{3}{2}+\epsilon}, \frac{3}{2}+\epsilon}(\Gamma) & \text{if} \quad 1 < p \leq \frac{3}{2} \\ W^{1-\frac{1}{p},p}(\Gamma) & \text{if} \quad p > \frac{3}{2} \end{cases}
$$

satisfying (0.2)-(0.3). Then the Stokes problem (S) has a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

Limiting cases

Now we discuss some limiting cases of the system

$$\begin{cases} -\Delta \mathbf{u}_\alpha + \nabla \pi_\alpha = \mathbf{f}, \quad \text{div } \mathbf{u}_\alpha = \mathbf{0} & \text{in } \Omega \\ \mathbf{u}_\alpha \cdot \mathbf{n} = 0, \quad 2\,[\mathbb{D}(\mathbf{u}_\alpha)\mathbf{n}]_\tau + \alpha\, \mathbf{u}_{\alpha\tau} = \mathbf{h} & \text{on } \Gamma \end{cases} \tag{0.17}$$

Theorem (α tends to 0)

Let $p \geq 2$ and $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ s.t. $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ, $\alpha \in L^{t(p)}(\Gamma)$ satisfying (0.2)-(0.3) and when Ω is axisymmetric, the following compatibility condition is assumed,

$$\langle \mathbf{f}, \beta \rangle_\Omega + \langle \mathbf{h}, \beta \rangle_\Gamma = 0\ . \tag{0.18}$$

Then as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$,

$$(\mathbf{u}_\alpha, \pi_\alpha) \to (\mathbf{u}_0, \pi_0) \quad \textit{in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

where $(\mathbf{u}_0, \pi_0)$ is a solution of the following system

$$
\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} = g, \quad 2\left[\mathbb{D}(\mathbf{u})\mathbf{n}\right]_{\tau} = \mathbf{h} & \text{on } \Gamma\,. \end{cases} \tag{0.19}
$$

Idea of the proof : Note that when Ω is axisymmetric, to expect the limiting system to be (0.19), we must assume the compatibility condition (0.18) since this is the necessary condition for the existence of solution of the system (0.19).

Let $\alpha \to 0$ in $L^{t(p)}(\Gamma)$ i.e. except the case when $\alpha \geq \alpha^* > 0$ in (0.3b). Now from the L^p-estimates, we get that $(\mathbf{u}_\alpha, \pi_\alpha)$ is bounded in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$. Then $\exists$ $(\mathbf{u}_0, \pi_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$
(\mathbf{u}_\alpha, \pi_\alpha) \rightharpoonup (\mathbf{u}_0, \pi_0) \quad \text{weakly in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega).
$$

It can be easily shown that $(\mathbf{u}_0, \pi_0)$ is the unique solution of the Stokes problem (0.19) with Navier boundary condition, corresponding to $\alpha = 0$.

Indeed, since $(\mathbf{u}_\alpha, \pi_\alpha)$ is the solution of (0.17), it satisfies the weak formulation (0.4). Now as explained before, $\mathbf{u}_\alpha \rightharpoonup \mathbf{u}_0$ in $\mathbf{W}^{1,p}(\Omega)$ implies

$$\mathbf{u}_{\alpha\tau} \rightharpoonup \mathbf{u}_{0\tau} \quad \text{in} \;\; \mathbf{L}^s(\Gamma)$$

and because $\alpha \to 0$ in $L^{t(p)}(\Gamma)$,

$$\alpha \mathbf{u}_{\alpha\tau} \rightharpoonup \mathbf{0} \quad \text{in} \;\; \mathbf{L}^{m'}(\Gamma).$$

Hence in the weak formulation (0.4), the boundary term in the left hand side goes to 0. So passing to the limit, $\mathbf{u}_0$ satisfies,

$$\forall\, \varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega), \quad 2\int_\Omega \mathbb{D}(\mathbf{u}_0) : \mathbb{D}(\varphi)\, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \varphi \, \mathrm{d}x + \langle \mathbf{h}, \varphi \rangle_\Gamma \; .$$

Now by taking difference between the system (0.17) and the limiting system (0.19), we get,

$$\begin{cases} -\Delta(\mathbf{u}_\alpha - \mathbf{u}_0) + \nabla(\pi_\alpha - \pi_0) = \mathbf{0}, \quad \mathrm{div}\,(\mathbf{u}_\alpha - \mathbf{u}_0) = 0 & \text{in} \;\; \Omega\;, \\ (\mathbf{u}_\alpha - \mathbf{u}_0) \cdot \mathbf{n} = 0, \quad 2\left[\mathbb{D}(\mathbf{u}_\alpha - \mathbf{u}_0)\mathbf{n}\right]_\tau + \alpha(\mathbf{u}_\alpha - \mathbf{u}_0)_\tau = -\alpha \mathbf{u}_{0\tau} & \text{on} \;\; \Gamma\;. \end{cases}$$

Once again using the usual L^p-estimates for the above system and also using Hölder inequality and trace theorem, we obtain

$$\begin{aligned} & \|\mathbf{u}_\alpha - \mathbf{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} \\ & \leq C(\Omega)\ \|\alpha \mathbf{u}_{0\tau}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} \\ & \leq C(\Omega)\ \|\alpha\|_{L^{t(p)}(\Gamma)} \|\mathbf{u}_0\|_{\mathbf{W}^{1,p}(\Omega)}\ . \end{aligned}$$

Hence, $\mathbf{u}_\alpha - \mathbf{u}_0$ and $\pi_\alpha - \pi_0$ both tend to zero in the same rate as α.

Theorem (α tends to ∞)

Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{h} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ *and* α *is a constant satisfying* (0.2)-(0.3). *As* $\alpha \to \infty$, *we have the convergence,*

$$(\mathbf{u}_\alpha, \pi_\alpha) \to (\mathbf{u}_\infty, \pi_\infty) \quad \textit{in} \quad \mathbf{H}^1(\Omega) \times L^2(\Omega)$$

where $(\mathbf{u}_\infty, \pi_\infty)$ *is the unique solution of the Stokes problem with Dirichlet boundary condition,*

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \textit{in} \quad \Omega\ , \\ \operatorname{div} \mathbf{u} = 0 & \textit{in} \quad \Omega\ , \\ \mathbf{u} = \mathbf{0} & \textit{on} \quad \Gamma\ . \end{cases}$$

Idea of the proof :

The proof is very much similar to the previous theorem. The key point is to write the system in the following way

$$\begin{cases} -\Delta \mathbf{u}_\alpha + \nabla \pi_\alpha = \mathbf{f} & \text{in} \quad \Omega \\ \text{div } \mathbf{u}_\alpha = 0 & \text{in} \quad \Omega \\ \mathbf{u}_\alpha = \frac{1}{\alpha}\left(\mathbf{h} - 2[\mathbb{D}(\mathbf{u}_\alpha)\mathbf{n}]_\tau\right) & \text{on} \quad \Gamma \ . \end{cases}$$

Then using the good H^2-estimate as we deduced before, we can show the existence of a weak limit which solves the Stokes system with Dirichlet boundary condition and finally taking the differences between the two systems, we can deduce the strong convergence.

Theorem (α less regular)

Let $\mathbf{f} \in \mathbf{L}^{\frac{6}{5}}(\Omega), \mathbf{h} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ *with* $\mathbf{h} \cdot \mathbf{n} = 0$ *on* Γ *and* $\alpha \in L^{\frac{4}{3}}(\Gamma)$ *satisfying* (0.2)-(0.3). *Then the Stokes problem* (S) *has a solution* $(\mathbf{u}, \pi)$ *in* $\mathbf{H}^1(\Omega) \times L^2(\Omega)$.

The above result can be proved using the density of $\mathcal{D}(\Gamma)$ in $L^{\frac{4}{3}}(\Gamma)$ and from the good estimates (independent of α) in $\mathbf{H}^1(\Omega)$.

Now we want to discuss the question, we posed before, about the dependence of the constant in the Inf-Sup condition (0.16).

Claim : Let Ω be $\mathcal{C}^{2,1}$. For $1 < p < \infty$ and for any $\alpha > 0$ constant, there exists $\gamma > 0$, independent of α such that

$$\underset{\substack{\varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega) \\ \varphi \neq 0}}{inf} \quad \underset{\substack{\mathbf{u} \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega) \\ \mathbf{u} \neq 0}}{sup} \frac{2\int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi)\, \mathrm{d}x + \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau \, \mathrm{d}s}{\|\mathbf{u}\|_{\mathbf{V}^{p}_{\sigma,\tau}(\Omega)} \, \|\varphi\|_{\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq \gamma.$$

Idea of the proof. First we claim that the constant $\gamma(\alpha)$ in (0.16) is decreasing with respect to α

$$\text{i.e.} \qquad \alpha_1 \leq \alpha_2 \quad \text{implies} \quad \gamma(\alpha_1) \geq \gamma(\alpha_2)$$

which implies directly our main claim ! The Inf-Sup condition (0.16) implies that $\exists$ a unique $\mathbf{u} \in \mathbf{V}^{p}_{\sigma,\tau}(\Omega)$ s.t. for any $\ell \in \left[\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)\right]'$,

$$A\mathbf{u} = \ell$$

and

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq \frac{1}{\gamma(\alpha)} \ \|\ell\|_{\left[\mathbf{V}^{p'}_{\sigma,\tau}(\Omega)\right]'} \tag{0.20}$$

where

$$\langle A\mathbf{u}, \varphi \rangle = 2 \int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi) \ \mathrm{d}x + \int_\Gamma \alpha \mathbf{u}_\tau \cdot \varphi_\tau \ \mathrm{d}s \qquad \forall \varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega).$$

Now for $p = 2$, choosing $\varphi = \mathbf{u}$, we get,

$$2 \int_\Omega |\mathbb{D}(\mathbf{u})|^2 \ \mathrm{d}x + \alpha \int_\Gamma |\mathbf{u}_\tau|^2 \ \mathrm{d}s = \langle \ell, \mathbf{u} \rangle \leq \|\ell\|_{\left[\mathbf{V}^2_{\sigma,\tau}(\Omega)\right]'} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}.$$

So for Ω not axisymmetric, we obtain,

$$2 \int_\Omega |\mathbb{D}(\mathbf{u})|^2 \ \mathrm{d}x + \alpha \int_\Gamma |\mathbf{u}_\tau|^2 \ \mathrm{d}s \leq C(\alpha) \ \|\ell\|_{\left[\mathbf{V}^2_{\sigma,\tau}(\Omega)\right]'} \|\mathbb{D}(\mathbf{u})\|_{\mathbf{L}^2(\Omega)}.$$

Thus comparing with the estimate (0.20), we can conclude that as α increases, $\frac{1}{\gamma(\alpha)}$ also has to increase and hence $\gamma(\alpha)$ decreases.

Non-linear problem

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \qquad 2\left[\mathbb{D}(\mathbf{u})\mathbf{n}\right]_{\tau} + \alpha\, \mathbf{u}_{\tau} = \mathbf{h} & \text{on } \Gamma. \end{cases} \tag{NS}$$

Let $1 < p < \infty$ and $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega), \mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ with $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ and $\alpha \in L^{t(p)}(\Gamma)$ satisfying (0.2)-(0.3). Then $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies (NS) in the sense of distribution is equivalent to:
$\mathbf{u} \in \mathbf{V}^p_{\sigma,\tau}(\Omega)$ such that for all $\varphi \in \mathbf{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2\int_{\Omega} \mathbb{D}(\mathbf{u}) : \mathbb{D}(\varphi)\, \mathrm{d}x + b(\mathbf{u}, \mathbf{u}, \varphi) + \int_{\Gamma} \alpha \mathbf{u}_{\tau} \cdot \varphi_{\tau}\, \mathrm{d}s = \int_{\Omega} \mathbf{f} \cdot \varphi\, \mathrm{d}x + \langle \mathbf{h}, \varphi \rangle_{\Gamma}. \tag{0.21}$$

where $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla)\, \mathbf{v} \cdot \mathbf{w}\, \mathrm{d}x$.

We first give some interesting properties of the operator b.

Lemma

The trilinear form b is defined and continuous on $\mathbf{V}^2_{\sigma,\tau}(\Omega) \times \mathbf{V}^2_{\sigma,\tau}(\Omega) \times \mathbf{V}^2_{\sigma,\tau}(\Omega)$*. Also, we have*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \mathbf{0} \tag{0.22}$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall\ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}^2_{\sigma,\tau}(\Omega)\ .$$

Also note that

$$b(\mathbf{u}, \mathbf{u}, \beta) = 0 \quad \text{and} \quad b(\beta, \beta, \mathbf{u}) = 0.$$

Now we give the existence and estimate of generalized solution of the Navier-Stokes problem (NS).

Theorem

Let $p \geq 2$ *and* $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ *with* $\mathbf{h} \cdot \mathbf{n} = 0$ *on* Γ, $\alpha \in L^{t(p)}(\Gamma)$ *satisfying* (0.2)-(0.3)*. Then the problem* (NS) *has a solution* $(\mathbf{u}, \pi)$ *belonging to* $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$*.*

Estimate.

In the Hilbert case, we have the following estimates as in the linear problem.
(I) if Ω is not axisymmetric, then

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

(II) if Ω is axisymmetric and

- $\alpha \geq \alpha_* > 0$ on $\Gamma_0 \subseteq \Gamma$, then

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

- $\mathbf{f}, \mathbf{h}$ satisfy the condition:

$$\int_\Omega \mathbf{f} \cdot \beta \, \mathrm{d}x + \langle \mathbf{h}, \beta \rangle_\Gamma = 0$$

then, the solution $\mathbf{u}$ satisfies $\int_\Gamma \alpha \mathbf{u} \cdot \beta \, \mathrm{d}s = 0$ and

$$\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} + \int_\Gamma \alpha |\mathbf{u}_\tau|^2 \, \mathrm{d}s \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right)^2.$$

Moreover, if α is a constant, then

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathbf{f}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \right).$$

Idea of the proof.

- $p = 2$. The existence of solution can be shown using standard arguments applying the Galerkin method and Brower's fixed point theorem.
- $p > 2$. We can get the existence of solution by using repetitively the regularity results of the Stokes system.
- And the estimates follows from the weak formulation exactly by the same argument as in the linear problem due to the property (0.22) of the trilinear form b.

Finally we give some results of the limiting problems corresponding to the non-linear system.

Theorem (α tends to 0)

Let $(\mathbf{u}_\alpha, \pi_\alpha)$ *be the solution of* (NS) *where*

$$\mathbf{f} \in \mathbf{L}^{6/5}(\Omega), \mathbf{h} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) \ s.t. \ \mathbf{h} \cdot \mathbf{n} = 0 \ on \ \Gamma, \alpha \in L^2(\Gamma).$$

Also, in addition, assume that when Ω *is axisymmetric,* α *is a constant and the following compatibility condition is satisfied,*

$$\langle \mathbf{f}, \beta \rangle_\Omega + \langle \mathbf{h}, \beta \rangle_\Gamma = 0 \ .$$

Then we have the convergence,

$$(\mathbf{u}_\alpha, \pi_\alpha) \rightarrow (\mathbf{u}_0, \pi_0) \quad in \quad \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega) \quad as \quad \alpha \rightarrow 0 \quad in \quad L^2(\Gamma)$$

where $(\mathbf{u}_0, \pi_0)$ *is a solution of the following Navier-Stokes problem*

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0 & in \quad \Omega \ , \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad 2 \left[\mathbb{D}(\mathbf{u}) \mathbf{n} \right]_\tau = \mathbf{h} & on \quad \Gamma \ . \end{cases}$$

Theorem (α tends to ∞)

Let $(\mathbf{u}_\alpha, \pi_\alpha)$ *be the solution of* (NS) *where*

$$\mathbf{f} \in \mathbf{L}^{\frac{6}{5}}(\Omega), \mathbf{h} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) \ \textit{s.t.} \ \mathbf{h} \cdot \mathbf{n} = 0 \ \textit{on} \ \Gamma \ \textit{and} \ \alpha \ \textit{is a constant} .$$

Then as $\alpha \to \infty$, *we have the convergence,*

$$(\mathbf{u}_\alpha, \pi_\alpha) \rightharpoonup (\mathbf{u}_\infty, \pi_\infty) \quad \textit{weakly in} \quad \mathbf{H}^1(\Omega) \times L^2(\Omega)$$

where $(\mathbf{u}_\infty, \pi_\infty)$ *is a solution of the Navier-Stokes problem with Dirichlet boundary condition,*

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f} & \textit{in} \ \ \Omega \\ \text{div } \mathbf{u} = 0 & \textit{in} \ \ \Omega \\ \mathbf{u} = \mathbf{0} & \textit{on} \ \ \Gamma \ . \end{cases} \tag{0.23}$$

If $\mathbf{f} \in \mathbf{L}^q(\Omega)$ *with* $q > \frac{6}{5}$, *then*

$$(\mathbf{u}_\alpha, \pi_\alpha) \to (\mathbf{u}_\infty, \pi_\infty) \quad \textit{in} \quad \mathbf{H}^1(\Omega) \times L^2(\Omega).$$

Thank You