Une classe d’inéquations variationnelles implicites et applications à des problèmes quasistatiques de contact

Marius Cocou$^1,2$

$^1$Laboratoire de Mécanique et d’Acoustique CNRS

$^2$Faculté des Sciences, Université d’Aix-Marseille

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Outline

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Quasistatic Contact Problems

- A. Signorini (1933, 1959), G. Fichera (1964, 1972), G. Duvaut and J.L. Lions (1972) - static elastic problems with unilateral contact
- J.J. Telega (1991) - variational formulation of quasistatic elastic problems with unilateral contact and Coulomb friction
- C. Eck, J. Jarušek, and J. Stará (2013) - normal compliance contact models with finite interpenetration.
- M. Sofonea and co-workers (1993,...) - contact problems with friction in viscoplasticity and viscoelasticity
- P.J. Rabier and O.V. Savin (2000) - an intermediate pointwise contact condition in the static case
- M.C. (2014, 2015) - dynamic contact with intermediate contact conditions between two viscoelastic bodies.
An implicit variational inequality
For simplicity and in view of applications to contact mechanics, we shall confine
attention to the case when $\Omega$ is an open, bounded, connected set $\Omega \subset \mathbb{R}^d$, $d = 2, 3,$
with the boundary $\Gamma \in C^{1,1}$ and with $\Xi$ an open part of $\Gamma$.
Denote $\Xi_T := \Xi \times (0, T)$, where $0 < T < +\infty$, and define

$$L^2_-(\Xi) := \{ \delta \in L^2(\Xi); \delta \leq 0 \text{ a.e. in } \Xi \},$$

$$L^2_-(\Xi_T) := \{ \delta \in L^2(0, T; L^2(\Xi)); \delta \leq 0 \text{ a.e. in } \Xi_T \}.$$

Let $\kappa, \overline{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ be two mappings with $\kappa$ lower semicontinuous and $\overline{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\kappa(s) \leq \overline{\kappa}(s) \leq 0 \quad \forall s \in \mathbb{R}, \quad (1)$$

$$\exists r_0 \geq 0 \text{ such that } |\kappa(s)| \leq r_0 \quad \forall s \in \mathbb{R}. \quad (2)$$

For every $\zeta \in L^2(\Xi)$, define the following nonempty subset of $L^2_-(\Xi)$:

$$\Lambda(\zeta) = \{ \eta \in L^2_-(\Xi); \kappa \circ \zeta \leq \eta \leq \overline{\kappa} \circ \zeta \text{ a.e. in } \Xi \}. \quad (3)$$

Since $\text{meas}(\Xi) < \infty$ and $\kappa, \overline{\kappa}$ satisfy (2), it is also readily seen that for all $\zeta \in L^2(\Xi)$ the set $\Lambda(\zeta)$ is bounded in norm in $L^2(\Xi)$ by $R_0 = r_0(\text{meas}(\Xi))^{1/2}$ and in $L^\infty(\Xi)$ by $r_0$. 
Let \((V, \|\cdot\|, \langle \cdot, \cdot \rangle)\) and \((U, \|\cdot\|_U)\) be two Hilbert spaces such that \(V \subset U\) with continuous and compact embedding. Consider a functional \(F : V \to \mathbb{R}\) differentiable on \(V\) and assume that its derivative \(F' : V \to V\) is strongly monotone and Lipschitz continuous, that is there exist two constants \(\alpha, \beta > 0\) such that for all \(u, v \in V\)

\[
\alpha\|v - u\|^2 \leq \langle F'(v) - F'(u), v - u \rangle
\] (4)

and

\[
\|F'(v) - F'(u)\| \leq \beta\|v - u\|.
\] (5)

It is easily seen that for all \(u, v \in V\) it results

\[
\langle F'(u), v - u \rangle + \frac{\alpha}{2}\|v - u\|^2 \leq F(v) - F(u)
\]

\[
\leq \langle F'(u), v - u \rangle + \frac{\beta}{2}\|v - u\|^2.
\] (6)

We remark that since \(F\) satisfies (6), it follows that \(F\) is strictly convex and sequentially weakly lower semicontinuous on \(V\).
Let $(X, \| \cdot \|_X)$ be a Hilbert space such that $X \subset L^2(\Gamma)$ with continuous and compact embedding, and $l_0 : V \to X$, $l : V \to L^2(\Xi)$, $\phi : L^2_-(\Xi) \times V \to \mathbb{R}$ satisfying

$$l_0 \text{ is linear and continuous,}$$

$$\exists k_1 > 0 \text{ such that } \forall v_1, v_2 \in V,$$

$$\|l(v_1) - l(v_2)\|_{L^2(\Xi)} \leq k_1 \|v_1 - v_2\|_U,$$

$$\forall \gamma, \delta \in L^2_-(\Xi), \forall v, w \in V \text{ verifying } \gamma \in \Lambda(l(v)) \text{ and } \delta \in \Lambda(l(w)),$$

$$\langle \gamma - \delta, l_0(v - w) \rangle_{L^2(\Xi)} \leq 0.$$
\[ \exists k_2, k_3 > 0 \text{ such that } \forall \gamma, \delta \in L_2^2(\Xi), \forall v \in V, \]
\[ |\phi(\gamma, v) - \phi(\delta, v)| \leq k_2 \|\gamma - \delta\|_{L_2^2(\Xi)} \|v\|_U, \quad (13) \]
\[ |\phi(\gamma, v) - \phi(\delta, v)| \leq k_3 \|\gamma - \delta\|_{X'} \|v\|, \quad (14) \]
\[ \exists k_4 > 0 \text{ such that } \|\gamma_1 - \gamma_2\|_{X'} \leq k_4(\|u_1 - u_2\| + \|f_1 - f_2\|), \quad (15) \]
for all \( \gamma_{1,2} \in L_2^2(\Xi), u_{1,2}, f_{1,2}, d_{1,2} \in V \) verifying

\[ (Q_1) \quad \langle F'(u_1), v - u_1 \rangle - \langle \gamma_1, l_0(v - u_1) \rangle_{L_2^2(\Xi)} + \phi(\gamma_1, v - d_1) - \phi(\gamma_1, u_1 - d_1) \geq \langle f_1, v - u_1 \rangle \quad \forall v \in V, \]

\[ (Q_2) \quad \langle F'(u_2), v - u_2 \rangle - \langle \gamma_2, l_0(v - u_2) \rangle_{L_2^2(\Xi)} + \phi(\gamma_2, v - d_2) - \phi(\gamma_2, u_2 - d_2) \geq \langle f_2, v - u_2 \rangle \quad \forall v \in V, \]

and we assume that \( k_3 k_4 < \alpha. \) \quad (16)
Let \( f \in W^{1,2}(0, T; V), u^0 \in V, \lambda^0 \in \Lambda(l(u^0)) \) be given and satisfy the following compatibility condition:

\[
\langle F'(u^0), \nu - u^0 \rangle - \langle \lambda^0, l_0(\nu - u^0) \rangle_{L^2(\Xi)} + \phi(\lambda^0, \nu) - \phi(\lambda^0, u^0) \geq \langle f(0), \nu - u^0 \rangle \quad \forall \nu \in V.
\]  

(17)

Consider the following problem.

**Problem Q**: Find \( u \in W^{1,2}(0, T; V), \lambda \in W^{1,2}(0, T; X') \) such that \( u(0) = u^0, \lambda(0) = \lambda^0, \lambda(t) \in \Lambda(l(u(t))) \) for almost all \( t \in (0, T) \), and

\[
\langle F'(u), \nu - \dot{u} \rangle - \langle \lambda, l_0(\nu - \dot{u}) \rangle_{L^2(\Xi)} + \phi(\lambda, \nu) \geq \langle f, \nu - \dot{u} \rangle \quad \forall \nu \in V \quad \text{a.e. on } (0, T).
\]  

(18)
For \( n \in \mathbb{N}^* \), we set \( \Delta t := T/n \), \( t_i := i \Delta t \), \( i = 0, 1, \ldots, n \). If \( \theta \) is a continuous function of \( t \in [0, T] \) valued in some vector space, we use the notations \( \theta^i := \theta(t_i) \) unless \( \theta = u \), and if \( \varpi^i, \quad \forall \ i \in \{0, 1, \ldots, n\} \), are elements of some vector space, then we set

\[
\partial \varpi^i := \frac{\varpi^{i+1} - \varpi^i}{\Delta t}, \quad \Delta \varpi^i := \varpi^{i+1} - \varpi^i \quad \forall \ i \in \{0, 1, \ldots, n - 1\}.
\]

We approximate the problem \( Q \) using the following sequence of incremental problems \((Q^{i,n})_{i=0,1,\ldots,n-1}\).

**Problem \( Q^{i,n} \)**: Find \( u^{i+1} \in V \), \( \lambda^{i+1} \in \Lambda(l(u^{i+1})) \) such that

\[
\langle F'(u^{i+1}), v - \partial u^i \rangle - \langle \lambda^{i+1}, l_0(v - \partial u^i) \rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v) - \phi(\lambda^{i+1}, \partial u^i) \geq \langle f^{i+1}, v - \partial u^i \rangle \quad \forall \ v \in V.
\] (19)

For all \( i \in \{0, 1, \ldots, n - 1\} \) the problem \( Q^{i,n} \) is equivalent to the following implicit variational inequality:

**Problem \( \hat{Q}^{i,n} \)**: Find \( u^{i+1} \in V \), \( \lambda^{i+1} \in \Lambda(l(u^{i+1})) \) such that

\[
\langle F'(u^{i+1}), v - u^{i+1} \rangle - \langle \lambda^{i+1}, l_0(v - u^{i+1}) \rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v - u^i) - \phi(\lambda^{i+1}, u^{i+1} - u^i) \geq \langle f^{i+1}, v - u^{i+1} \rangle \quad \forall \ v \in V.
\] (20)
Let us define the following functions:

\( u_n(0) = \hat{u}_n(0) = u^0, \ \lambda_n(0) = \lambda^0, \ f_n(0) = f^0 \) and

\[ \forall i \in \{0, 1, \ldots, n-1\}, \ \forall t \in (t_i, t_{i+1}], \]

\( u_n(t) = u^{i+1}, \ \lambda_n(t) = \lambda^{i+1}, \)

\( \hat{u}_n(t) = u^i + (t - t_i)\partial u^i, \)

\( \hat{\lambda}_n(t) = \lambda^i + (t - t_i)\partial \lambda^i, \)

\( f_n(t) = f^{i+1}. \)

Then for all \( n \in N^* \) each of the sequences of inequalities \((Q^i,n)_{i=0,1,\ldots,n-1}, \)

\( (\hat{Q}^i,n)_{i=0,1,\ldots,n-1} \) is equivalent to the following incremental formulation.

**Problem \( Q^n \):** Find \( u_n \in L^2(0, T; V), \ \lambda_n \in L^2(\Xi_T) \) such that \( \lambda_n(t) \in \Lambda(l(u_n(t))) \)

\[ \forall t \in (0, T) \) and

\[
\langle F'(u_n(t)), v - \frac{d}{dt}\hat{u}_n(t) \rangle - \langle \lambda_n(t), l_0(v - \frac{d}{dt}\hat{u}_n(t)) \rangle_{L^2(\Xi)}
\]

\[ + \phi(\lambda_n(t), v) - \phi(\lambda_n(t), \frac{d}{dt}\hat{u}_n(t)) \geq \langle f_n(t), v - \frac{d}{dt}\hat{u}_n(t) \rangle \forall v \in V, \ a.e. \ on \ (0, T). \]
We prove the existence of a fixed point of the multifunction $\Phi^i$ by using a corollary of the Ky Fan's fixed point theorem in the particular case of a reflexive Banach space.

**Definition**

Let $Y$ be a reflexive Banach space, $D$ a weakly closed set in $Y$, and $\Phi : D \to 2^Y \setminus \{\emptyset\}$ be a multivalued function. $\Phi$ is called sequentially weakly upper semicontinuous if $z_p \rightharpoonup z$, $y_p \in \Phi(z_p)$ and $y_p \rightharpoonup y$ imply $y \in \Phi(z)$.

**Proposition**

Let $Y$ be a reflexive Banach space, $D$ a convex, closed and bounded set in $Y$, and $\Phi : D \to 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $\Phi(z)$ is convex for every $z \in D$. Then $\Phi$ has a fixed point.

**Theorem**

Assume that (1 - 5), (7 - 14) hold. Then there exists $\lambda \in L^2_-(\Xi)$ such that $\lambda \in \Phi^i(\lambda)$ and $(u^{i+1}, \lambda^{i+1}) = (u_\lambda, \lambda)$ is a solution of the problem $\hat{Q}^{i,n}$. 
We now establish some useful estimates independent of $n$ for the solutions of the incremental formulations $\hat{Q}^i, n$ and $Q^n$.

Lemma

Under the above hypotheses, for all $n \in N^*$ and all $i \in \{0, 1, \ldots, n - 1\}$ the following estimates hold:

\[
\|u^{i+1}\| \leq C_1(\|\lambda^{i+1}\|_{X'} + \|f^{i+1}\| + \|F'(0)\|),
\]

\[
\|\Delta u^i\| \leq \frac{k_3}{\alpha} \|\Delta \lambda^i\|_{X'} + \frac{1}{\alpha} \|\Delta f^i\|),
\]

\[
\|\Delta \lambda^i\|_{X'} \leq k_4(\|\Delta u^i\| + \|\Delta f^i\|),
\]

\[
\|\Delta u^i\| \leq C_2 \|\Delta f^i\|,
\]

\[
\|\Delta \lambda^i\|_{X'} \leq C_3 \|\Delta f^i\|,
\]

where $C_2 = \frac{k_3 k_4 + 1}{\alpha - k_3 k_4}$, $C_3 = \frac{(\alpha + 1)k_4}{\alpha - k_3 k_4}$. 

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Lemma

For all \( n \in N^* \)

\[
\|u_n(t)\| \leq C_1 (\|\lambda_n(t)\|_{X'} + \|f_n(t)\| + \|F'(0)\|) \quad \forall \ t \in [0, T],
\] (27)

\[
\|u_n(t) - \hat{u}_n(t)\| \leq \frac{T}{n} \left\| \frac{d}{dt} \hat{u}_n(t) \right\| \leq C_2 \left\| f_n(t) - f_n(t - \frac{T}{n}) \right\| \leq C_2 \int_{t - \frac{T}{n}}^{\min\{t + \frac{T}{n}, T\}} \|\dot{f}(\tau)\| \, d\tau \quad \forall \ t \in [0, T],
\] (28)

\[
\|\lambda_n(t) - \hat{\lambda}_n(t)\|_{X'} \leq \frac{T}{n} \left\| \frac{d}{dt} \hat{\lambda}_n(t) \right\|_{X'} \leq C_3 \left\| f_n(t) - f_n(t - \frac{T}{n}) \right\| \quad \forall \ t \in [0, T],
\] (29)

\[
\|u_n - \hat{u}_n\|_{L^2(0,T;V)} = \frac{T}{n \sqrt{3}} \left\| \frac{d}{dt} \hat{u}_n \right\|_{L^2(0,T;V)} \leq C_2 \frac{T}{n \sqrt{3}} \|\dot{f}\|_{L^2(0,T;V)}.
\] (30)

\[
\|\lambda_n - \hat{\lambda}_n\|_{L^2(0,T;X')} = \frac{T}{n \sqrt{3}} \left\| \frac{d}{dt} \hat{\lambda}_n \right\|_{L^2(0,T;X')} \leq C_3 \frac{T}{n \sqrt{3}} \|\dot{f}\|_{L^2(0,T;V)}.
\] (31)
Lemma

There exist subsequences of \((u_n, \hat{u}_n)_n\) and \((\lambda_n, \hat{\lambda}_n)_n\), denoted by \((u_{n_p}, \hat{u}_{n_p})_p\) and \((\lambda_{n_p}, \hat{\lambda}_{n_p})_p\), respectively, and two elements \(u \in W^{1,2}(0, T; V)\), \(\lambda \in W^{1,2}(0, T; X') \cap L^2(\Xi_T)\) such that

\[ u_{n_p}(t) \rightharpoonup u(t) \text{ in } V \quad \forall \ t \in [0, T], \]  
\[ \hat{u}_{n_p} \rightharpoonup u \text{ in } W^{1,2}(0, T; V), \]  
\[ \lambda_{n_p}(t) \rightharpoonup \lambda(t) \text{ in } X' \quad \forall \ t \in [0, T], \]  
\[ \lambda_{n_p}, \hat{\lambda}_{n_p} \rightharpoonup \lambda \text{ in } L^2(0, T; L^2(\Xi)), \]  
\[ \hat{\lambda}_{n_p} \rightharpoonup \lambda \text{ in } W^{1,2}(0, T; X'), \]  
\[ \liminf_{p \to \infty} \int_0^T \phi(\lambda_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) \, dt \geq \int_0^T \phi(\lambda(t), \frac{d}{dt} \hat{u}(t)) \, dt. \]
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**Theorem**

Under the assumptions (1 - 5), (7 - 16), every convergent subsequence of the previous lemma, \((u_{np}, \hat{u}_{np})_p, (\lambda_{np}, \hat{\lambda}_{np})_p\), and their limits \(u \in W^{1,2}(0, T; V), \lambda \in W^{1,2}(0, T; X') \cap L^2(\Xi_T)\) have the following strong convergence properties

\[
 u_{np}(t) \to u(t) \quad \forall \ t \in [0, T], 
\]

\[
 \lambda_{np}(t) \to \lambda(t) \quad \forall \ t \in [0, T], 
\]

and \((u, \lambda)\) is a solution to the problem \(Q\).

**Proof.** The sequence \((\hat{Q}^{i,n})_{i=0,1,...,n-1}\) implies that for every \(t \in [0, T]\)

\[
 \langle F'(u_n(t)), v - u_n(t) \rangle - \langle \lambda_n(t), l_0(v - u_n(t)) \rangle_{L^2(\Xi)} 
+ \phi(\lambda_n(t), v - u_n(t)) \geq \langle f_n(t), v - u_n(t) \rangle \quad \forall \ v \in V, 
\]

and taking \(v = u\), by (6) we derive

\[
 F(u(t)) - F(u_{np}(t)) - \langle \lambda_{np}(t), l_0(u(t) - u_{np}(t)) \rangle_{L^2(\Xi)} 
+ \phi(\lambda_{np}(t), u(t) - u_{np}(t)) \geq \langle f_{np}(t), u(t) - u_{np}(t) \rangle + \frac{\alpha}{2} \|u(t) - u_{np}(t)\|^2 \quad \forall \ p \in \mathbb{N}. 
\]
Using that $F$ is sequentially weakly lower semicontinuous, (7), (13), the compact embeddings $X \subset L^2(\Xi), V \subset U$ and that for all $t \in [0, T]$ $(\lambda_{np}(t))_p$ is bounded in $L^2(\Xi)$ by $R_0$, the previous relation implies

$$
\limsup_{p \to \infty} \frac{\alpha}{2} \|u(t) - u_{np}(t)\|^2
\leq F(u(t)) + \limsup_{p \to \infty} (-F(u_{np}(t))) + \lim_{p \to \infty} |\langle \lambda_{np}(t), l_0(u(t) - u_{np}(t)) \rangle_{L^2(\Xi)}|
+ \lim_{p \to \infty} \phi(\lambda_{np}(t), u(t) - u_{np}(t)) - \lim_{p \to \infty} \langle f_{np}(t), u(t) - u_{np}(t) \rangle
\leq F(u(t)) - \liminf_{p \to \infty} F(u_{np}(t)) + \lim_{p \to \infty} \|\lambda_{np}(t)\|_{L^2(\Xi)} \|l_0(u(t) - u_{np}(t))\|_{L^2(\Xi)}
+ \lim_{p \to \infty} k_2 \|\lambda_{np}(t)\|_{L^2(\Xi)} \|u(t) - u_{np}(t)\|_U - \lim_{p \to \infty} \langle f_{np}(t), u(t) - u_{np}(t) \rangle
= F(u(t)) - \liminf_{p \to \infty} F(u_{np}(t)) \leq 0,
$$

which proves (38).
Now, we shall use the following compactness theorem proved by Simon (1987).

**Theorem**

Let $\hat{X}$, $\hat{U}$ and $\hat{Y}$ be three Banach spaces such that $\hat{X} \subset \hat{U} \subset \hat{Y}$ with compact embedding from $\hat{X}$ into $\hat{U}$.

(i) Let $G$ be bounded in $L^p(0, T; \hat{X})$, where $1 \leq p < \infty$, and $\partial G / \partial t := \{\dot{f}; \, f \in G\}$ be bounded in $L^1(0, T; \hat{Y})$. Then $G$ is relatively compact in $L^p(0, T; \hat{U})$.

(ii) Let $G$ be bounded in $L^\infty(0, T; \hat{X})$ and $\partial G / \partial t$ be bounded in $L^r(0, T; \hat{Y})$, where $r > 1$. Then $G$ is relatively compact in $C([0, T]; \hat{U})$.

By the Simon’s theorem with $G = (\hat{\lambda}_{np})_p$, $\hat{X} = L^2(\Xi)$, $\hat{U} = H^{\nu-1/2}(\Xi)$, $\hat{Y} = X'$, $r = 2$, $0 < \nu < \frac{1}{2}$, it follows that

$$\hat{\lambda}_{np} \rightarrow \lambda \text{ in } C([0, T]; X'),$$

so that by (29) we obtain (39).

It remains to prove that $(u, \lambda)$ is a solution of the problem $Q$. 
First, since \( \lambda_{np}(t) \in \Lambda(l(u_{np}(t))) \) for all \( t \in (0, T) \), we have

\[
\int_\omega \kappa \circ l(u_{np}) \leq \int_\omega \lambda_{np} \leq \int_\omega \bar{\kappa} \circ l(u_{np}),
\]

for every measurable subset \( \omega \subset \Xi_T \) and for all \( p \in \mathbb{N} \).

Using (38), (8), the semi-continuity of \( \kappa \) and \( \bar{\kappa} \), the relations (1), (2), (35), which implies the convergence property \( \int_\omega \lambda_{np} \to \int_\omega \lambda \), and passing to limits according to Fatou’s lemma, we obtain

\[
\int_\omega \kappa \circ l(u) \leq \int_\omega \lambda \leq \int_\omega \bar{\kappa} \circ l(u),
\]

for every measurable subset \( \omega \subset \Xi_T \), which implies \( \lambda(t) \in \Lambda(l(u(t))) \) for almost all \( t \in (0, T) \).

Second, integrating both sides in (21) over \([0, T]\) and passing to the limit, by the relations (38), (39), (33), (37), it follows that for all \( v \in L^2(0, T; V) \)

\[
\int_0^T \langle F'(u(t)), v(t) - \dot{u}(t) \rangle dt - \int_0^T \langle \lambda(t), l_0(v(t) - \dot{u}(t)) \rangle_{L^2(\Xi)} dt
\]

\[
+ \int_0^T \phi(\lambda(t), v(t)) dt - \int_0^T \phi(\lambda(t), \dot{u}(t)) dt \geq \int_0^T \langle f(t), v(t) - \dot{u}(t) \rangle dt.
\]

By Lebesgue’s theorem, it follows that \((u, \lambda)\) is a solution of the problem \(Q\).
Consider an elastic body occupying the set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ are open, disjoint parts of $\Gamma$ and $\text{meas}(\Gamma_1) > 0$. Assume the small deformation hypothesis and that the inertial effects are negligible. Denote by $u = u(x, t)$ the displacement field, by $\varepsilon$ the infinitesimal strain tensor and by $\sigma$ the stress tensor, with the components $u = (u_i)$, $\varepsilon = (\varepsilon_{ij})$ and $\sigma = (\sigma_{ij})$, respectively. We use the classical decompositions $u = u_N n + u_T$, $u_N = u \cdot n$, $\sigma n = \sigma_N n + \sigma_T$, $\sigma_N = (\sigma n) \cdot n$, where $n$ is the outward normal unit vector to $\Gamma$ with the components $n = (n_i)$. The usual summation convention will be used for $i, j, k, l = 1, \ldots, d$.

Consider the Hilbert space $V$ and the closed convex sets $L^2_-(\Gamma_3)$, $\Lambda_1(\zeta)$ as follows:

$$V = \{ v \in H^1(\Omega; \mathbb{R}^d); \ v = 0 \text{ a.e. on } \Gamma_1 \} ,$$

$$L^2_-(\Gamma_3) := \{ \delta \in L^2(\Gamma_3); \ \delta \leq 0 \ \text{a.e. in } \Gamma_3 \} ,$$

$$\Lambda_1(\zeta) = \{ \eta \in L^2_- (\Gamma_3); \ \kappa \circ \zeta \leq \eta \leq \bar{\kappa} \circ \zeta \ \text{ a.e. in } \Gamma_3 \} \ \forall \zeta \in L^2(\Gamma_3).$$

Assume that in $\Omega$ a body force $\varphi_1 \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^d))$ is prescribed, on $\Gamma_1$ the displacement vector equals zero and on $\Gamma_2$ a traction $\varphi_2 \in W^{1,2}(0, T; L^2(\Gamma_2; \mathbb{R}^d))$ is applied.
On $\Gamma_3$, the contact between the body and a support is possible with the initial gap denoted by $g_0$ and the gap corresponding to the solution $u$ denoted by $[u_N] := u_N - g_0$. We assume that there exists $g \in V$ such that $g_N = g_0$ on $\Gamma_3$. On the potential contact surface $\Gamma_3$, the displacements and the stress vector will satisfy some contact conditions having the following form:

$$\kappa([u_N]) \leq \sigma_N \leq \kappa([u_N]).$$

Assume that, for all $\gamma, \delta \in L^2(\Gamma_3)$ and all $v, w \in V$ such that $\gamma \in \Lambda_1([v_N])$, $\delta \in \Lambda_1([w_N]),$

$$\langle \gamma - \delta, v_N - w_N \rangle_{L^2(\Gamma_3)} \leq 0. \quad (45)$$

**Example 1.** *(Friction conditions with controlled normal stress)*

Let $M \geq 0$ be a constant and define

$$\kappa(s) = \kappa_M(s) = \begin{cases} 0 & \text{if } s < 0, \\ -M & \text{if } s \geq 0, \end{cases} \quad \bar{\kappa}(s) = \bar{\kappa}_M(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ -M & \text{if } s > 0. \end{cases}$$

The classical Signorini’s conditions correspond formally to $M = +\infty$.

**Example 2.** *(Normal compliance conditions)*

Various normal compliance conditions and friction laws can be obtained if one considers $\kappa = \bar{\kappa} = \kappa$, where $\kappa : \mathbb{R} \to \mathbb{R}$ is some negative, decreasing, and bounded Lipschitz continuous function, so that $\sigma_N$ is given by the relation $\sigma_N = \kappa([u_N])$.

It is easily seen that these two examples verify the condition (45).
Let $\mathcal{F} \geq 0$ be the coefficient of friction, assumed to be a Lipschitz continuous function on $\Gamma$, which ensures to belong to the set of the multipliers on $H^{1/2}(\Gamma)$ denoted by $\mathcal{M}$. Therefore the mapping $H^{1/2}(\Gamma) \ni v \mapsto \mathcal{F}v \in H^{1/2}(\Gamma)$ is bounded with norm $\|\mathcal{F}\|_\mathcal{M}$.

In order to describe the frictional contact conditions on $\Gamma_3$, we define

\[
\forall l \in V, \quad S_l := \{v \in V; \int_\Omega \sigma(v) \cdot \varepsilon(\psi)dx = \langle l, \psi \rangle_V \quad \forall \psi \in V \text{ such that } \psi = 0 \text{ a.e. on } \Gamma_3\}, \]

\[
L \in V, \quad \langle L, w \rangle_V = \langle \varphi_1, w \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle \varphi_2, w \rangle_{L^2(\Gamma_2; \mathbb{R}^d)} \quad \forall w \in V, \]

\[
\forall v \in S_L, \quad \langle \sigma_N(v), w \rangle_\Gamma = \int_\Omega \sigma(v) \cdot \varepsilon(\bar{w})dx - \langle L, \bar{w} \rangle_V \quad \forall w \in H^{1/2}(\Gamma),
\]

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, $\bar{w} \in V$ satisfies $\bar{w}_T = 0$ a.e. on $\Gamma_3$, $\bar{w}_N = w$ a.e. on $\Gamma_3$. It is easy to verify that for all $v \in S_L$ $\sigma_N(v)$ depends only on the values of $w$ on $\Gamma_3$ and not on the choices of $\bar{w}$ having the above properties.
A contact problem for a nonlinear Hencky material

Assume that the elastic body satisfies the following nonlinear Hencky-Mises constitutive equation:

\[ \sigma(u) = \hat{\sigma}(u) = (k - \frac{2}{3} \mu(\hat{\gamma}(u)))(\text{tr} \ \varepsilon(u)) I + 2 \mu(\hat{\gamma}(u)) \varepsilon(u), \]

where \( k \) is the constant bulk modulus, \( \mu \) is a continuously differentiable function in \([0, +\infty)\) satisfying

\[ 0 < \mu_0 \leq \mu(r) \leq \frac{3}{2} k, \quad 0 < \mu_1 \leq \mu(r) + 2 \frac{\partial \mu(r)}{\partial r} r \leq \mu_2, \quad \forall \ r \geq 0, \quad (46) \]

and, for all \( u, v \in V \),

\[ \hat{\gamma}(u) := \hat{\gamma}(u, u), \quad \hat{\gamma}(u, v) = -\frac{2}{3} \vartheta(u) \vartheta(v) + 2 \varepsilon(u) \cdot \varepsilon(v), \quad \vartheta(u) := \text{tr} \ \varepsilon(u) = \text{div} \ u. \]
Consider the following quasistatic contact problem with Coulomb friction.

**Problem** $P_1^c$ : Find $u$ such that $u(0) = u_0$ and, for all $t \in (0, T)$,

\[
\text{div } \sigma(u) = -\varphi_1 \quad \text{in } \Omega, \\
\sigma(u) = \hat{\sigma}(u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \Gamma_1, \quad \sigma n = \varphi_2 \quad \text{on } \Gamma_2, \\
\kappa([u_N]) \leq \sigma_N \leq \bar{\kappa}([u_N]) \quad \text{on } \Gamma_3, \\
|\sigma_T| \leq F|\sigma_N| \quad \text{and} \\
\dot{u}_T \neq 0 \Rightarrow \sigma_T = -F|\sigma_N|\frac{\dot{u}_T}{|\dot{u}_T|} \quad \text{on } \Gamma_3.
\]

Let $F_1 : V \rightarrow \mathbb{R}$ be defined by

\[
F_1(v) = \frac{1}{2} k \int_{\Omega} \vartheta^2(v) dx + \frac{1}{2} \int_{\Omega} \left( \int_0^{\hat{\gamma}(v)} \mu(r) dr \right) dx \quad \forall v \in V,
\]

and $J : L^2_-(\Gamma_3) \times V \rightarrow \mathbb{R}$ be defined by

\[
J(\gamma, v) = -\int_{\Gamma_3} F\gamma|v_T|ds \quad \forall \gamma \in L^2_-(\Gamma_3), \forall v \in V.
\]
An implicit variational inequality
Applications to two quasistatic contact problems
Perspectives

A contact problem for a nonlinear Hencky material
A contact problem for a linearly elastic body

\( F_1 \) is differentiable on \( V \) and for all \( u, v \in V \)

\[
\langle F'_1(u), v \rangle_v = \int_{\Omega} [(k - \frac{2}{3} \mu(\gamma(u))) \vartheta(u) \vartheta(v) + 2 \mu(\gamma(u)) \varepsilon(u) \cdot \varepsilon(v)] dx. \tag{54}
\]

Let \( u_0 \in V, \lambda^0 \in \Lambda_1([u_0N]) \) satisfy the following compatibility condition:

\[
\langle F'_1(u_0), v - u_0 \rangle_v - \langle \lambda^0, v_N - u_{0N} \rangle_{L^2(\Gamma_3)}
+ J(\lambda^0, v) - J(\lambda^0, u_0) \geq \langle L(0), v - u_0 \rangle_v \quad \forall v \in V. \tag{55}
\]

We have the following variational formulation of problem \( P^c_1 \).

**Problem \( P^c_1 \):** Find \( u \in W^{1,2}(0, T; V), \lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma)) \) such that \( u(0) = u_0, \lambda(0) = \lambda^0, \lambda(t) \in \Lambda_1([u_N(t)]) \) for almost all \( t \in (0, T) \), and

\[
\langle F'_1(u), v - \dot{u} \rangle_v - \langle \lambda, v_N - \dot{u}_N \rangle_{L^2(\Gamma_3)} + J(\lambda, v)
- J(\lambda, \dot{u}) \geq \langle L, v - \dot{u} \rangle_v \quad \forall v \in V \quad \text{a.e. on} \ (0, T). \tag{56}
\]
The Lagrange multiplier $\lambda \in L^2(\Gamma_3)$ satisfies the relation $\sigma_N = \lambda$ in $H^{-1/2}(\Gamma)$ that is

$$\langle \sigma_N(u), w \rangle_{\Gamma} = \langle \lambda, w \rangle_{L^2(\Gamma_3)} \quad \forall w \in H^{1/2}(\Gamma).$$

Taking $\Xi = \Gamma_3$, $\Lambda = \Lambda_1$, $V = V$, $U = H^\iota(\Omega; \mathbb{R}^d)$, $1 > \iota > \frac{1}{2}$, $X = H^{1/2}(\Gamma)$, $F = F_1$, $\phi = J$, $f = L$, and $l_0(v) = \nu_N$, $l(v) = [\nu_N] = \nu_N - g_0$ $\forall v \in V$, it results that the problem $P_1^x$ is a particular case of problem $Q$.

It is straightforward to verify the assumptions (1 - 5), (7 - 15), and also (16) if $\|F\|_M$ is sufficiently small, so that by the main theorem we obtain the following existence result.

**Proposition**

*Under the previous assumptions and if $\|F\|_M$ is sufficiently small there exists a solution to problem $P_1^x$.***
Let $A$ denote the elasticity tensor, with the components $A = (A_{ijkl})$ satisfying the following classical symmetry and ellipticity conditions: $A_{ijkl} = A_{jikl} = A_{klij} \in L^\infty(\Omega)$, $\forall i, j, k, l = 1, \ldots, d$, $\exists \alpha_A > 0$ such that $A_{ijkl} \tau_{ij} \tau_{kl} \geq \alpha_A \tau_{ij} \tau_{ij}$ $\forall \tau = (\tau_{ij})$ satisfying $\tau_{ij} = \tau_{ji}$, $\forall i, j = 1, \ldots, d$.

Consider the following elastic contact problem with Coulomb friction.

**Problem $P_2^c$:** Find $u$ such that $u(0) = u_0$, satisfying

$$\sigma(u) = A\varepsilon(u) \text{ in } \Omega,$$

and (47), (49 - 51) for all $t \in (0, T)$.

Let us define the bilinear and symmetric mapping $a : V \times V \to \mathbb{R}$ by

$$a(v, w) = \int_{\Omega} A\varepsilon(v) \cdot \varepsilon(w) \, dx = \int_{\Omega} A_{ijkl} \varepsilon_{ij}(v) \varepsilon_{kl}(w) \, dx \quad \forall \, v, w \in V.$$  

The form $a$ is continuous on $V \times V$ and, since $\text{meas}(\Gamma_1) > 0$, by Korn’s inequality is also $V$-elliptic.
Let \( u_0 \in V, \lambda^0 \in \Lambda_1([u_{0N}]) \) satisfy the following compatibility condition:

\[
a(u_0, v - u_0) - \langle \lambda^0, v_N - u_{0N} \rangle_{L^2(\Gamma_3)} + J(\lambda^0, v) - J(\lambda^0, u_0) \geq \langle L(0), v - u_0 \rangle_V \quad \forall v \in V. \tag{59}
\]

We have the following variational formulation of problem \( P_2^v \).

**Problem \( P_2^v \):** Find \( u \in W^{1,2}(0, T; V), \lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma)) \) such that \( u(0) = u_0, \lambda(0) = \lambda^0, \lambda(t) \in \Lambda_1([u_N(t)]) \) for almost all \( t \in (0, T) \), and

\[
a(u, v - \dot{u}) - \langle \lambda, v_N - \dot{u}_N \rangle_{L^2(\Gamma_3)} + J(\lambda, v) - J(\lambda, \dot{u}) \geq \langle L, v - \dot{u} \rangle_V \quad \forall v \in V \quad \text{a.e. on } (0, T). \tag{60}
\]

The Lagrange multiplier \( \lambda \in L^2(\Gamma_3) \) satisfies again the relation \( \sigma_N = \lambda \) in \( H^{-1/2}(\Gamma) \).

Taking \( \Xi, \Lambda, V, U, X, \phi, f, l_0, l \) as previously and \( F(v) = \frac{1}{2} a(v, v) \) \( \forall v \in V \), we see that the problem \( P_2^v \) is a particular case of problem \( Q \) so that by using again the existence theorem one obtains the following existence result.

**Proposition**

*Under the previous assumptions and if \( ||F||_M \) is sufficiently small there exists a solution to problem \( P_2^v \).*
Perspectives

- Nonlinear quasistatic and dynamic unilateral contact problems with friction
- More complex contact interaction laws (in quasistatic or dynamic cases)
- Numerical analysis and solution methods