Une classe d'inéquations variationnelles implicites et applications à des problèmes quasistatiques de contact

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Quasistatic Contact Problems

- A. Signorini (1933, 1959), G. Fichera (1964, 1972) , G. Duvaut and J.L. Lions (1972) - static elastic problems with unilateral contact
- J. Nečas, J. Jarušek, J. Haslinger (1980), J. Jarušek (1983) - unilateral contact with local Coulomb friction
- J. Martins and J. T. Oden (1985, 1987) - normal compliance laws
- J.J. Telega (1991) - variational formulation of quasistatic elastic problems with unilateral contact and Coulomb friction
- L.E. Andersson (2000), M.C. and Rocca (2000, 2001) - mathematical analysis of quasistatic elastic problems with unilateral contact and local Coulomb friction
- C. Eck, J. Jarušek, and J. Stará (2013) - normal compliance contact models with finite interpenetration.
- M. Sofonea and co-workers (1993,...) - contact problems with friction in viscoplasticity and viscoelasticity
- P.J. Rabier and O.V. Savin (2000) - an intermediate pointwise contact condition in the static case
- M.C. (2014, 2015) - dynamic contact with intermediate contact conditions between two viscoelastic bodies.

An implicit variational inequality

For simplicity and in view of applications to contact mechanics, we shall confine attention to the case when Ω is an open, bounded, connected set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with the boundary $\Gamma \in C^{1,1}$ and with Ξ an open part of Γ.
Denote $\Xi_T := \Xi \times (0, T)$, where $0 < T < +\infty$, and define

$$L^2_-(\Xi) := \{\delta \in L^2(\Xi);\ \delta \leq 0 \ \text{ a.e. in } \ \Xi\},$$

$$L^2_-(\Xi_T) := \{\delta \in L^2(0, T; L^2(\Xi));\ \delta \leq 0 \ \text{ a.e. in } \ \Xi_T\}.$$

Let $\underline{\kappa},\ \overline{\kappa} : \mathbb{R} \to \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\overline{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\underline{\kappa}(s) \leq \overline{\kappa}(s) \leq 0 \quad \forall\, s \in \mathbb{R}, \tag{1}$$

$$\exists\, r_0 \geq 0 \text{ such that } |\underline{\kappa}(s)| \leq r_0 \quad \forall\, s \in \mathbb{R}. \tag{2}$$

For every $\zeta \in L^2(\Xi)$, define the following nonempty subset of $L^2_-(\Xi)$:

$$\Lambda(\zeta) = \{\eta \in L^2_-(\Xi); \underline{\kappa} \circ \zeta \leq \eta \leq \overline{\kappa} \circ \zeta \ \text{ a.e. in } \ \Xi\}. \tag{3}$$

Since $\text{meas}(\Xi) < \infty$ and $\underline{\kappa}$, $\overline{\kappa}$ satisfy (2), it is also readily seen that for all $\zeta \in L^2(\Xi)$ the set $\Lambda(\zeta)$ is bounded in norm in $L^2(\Xi)$ by $R_0 = r_0(\text{meas}(\Xi))^{1/2}$ and in $L^\infty(\Xi)$ by r_0.

Let $(V,\ \|.\|,\ \langle .\, ,.\rangle)$ and $(U,\ \|.\|_U)$ be two Hilbert spaces such that $V \subset U$ with continuous and compact embedding.
Consider a functional $F : V \to \mathbb{R}$ differentiable on V and assume that its derivative $F' : V \to V$ is strongly monotone and Lipschitz continuous, that is there exist two constants $\alpha,\ \beta > 0$ such that for all $u, v \in V$

$$\alpha\|v-u\|^2 \leq \langle F'(v) - F'(u), v-u\rangle \tag{4}$$

and

$$\|F'(v) - F'(u)\| \leq \beta\|v-u\|. \tag{5}$$

It is easily seen that for all $u, v \in V$ it results

$$\begin{aligned}\langle F'(u), v-u\rangle + \frac{\alpha}{2}\|v-u\|^2 &\leq F(v) - F(u)\\ &\leq \langle F'(u), v-u\rangle + \frac{\beta}{2}\|v-u\|^2.\end{aligned} \tag{6}$$

We remark that since F satisfies (6), it follows that F is strictly convex and sequentially weakly lower semicontinuous on V.

Let $(X, \|.\|_X)$ be a Hilbert space such that $X \subset L^2(\Gamma)$ with continuous and compact embedding, and $l_0 : V \to X,\ l : V \to L^2(\Xi),\ \phi : L^2_-(\Xi) \times V \to \mathbb{R}$ satisfying

$$l_0 \text{ is linear and continuous,} \tag{7}$$

$$\begin{aligned}&\exists\, k_1 > 0 \text{ such that } \forall\, v_1, v_2 \in V,\\ &\|l(v_1) - l(v_2)\|_{L^2(\Xi)} \leq k_1 \|v_1 - v_2\|_U,\end{aligned} \tag{8}$$

$$\begin{aligned}&\forall\, \gamma, \delta \in L^2_-(\Xi), \forall\, v, w \in V \text{ verifying } \gamma \in \Lambda(l(v)) \text{ and } \delta \in \Lambda(l(w)),\\ &\langle \gamma - \delta, l_0(v - w)\rangle_{L^2(\Xi)} \leq 0.\end{aligned} \tag{9}$$

$$\forall\, \gamma \in L^2_-(\Xi),\ \forall\, \theta \geq 0,\ \forall\, v_1,\ v_2,\ v \in V,$$

$$\phi(\gamma, v_1 + v_2) \leq \phi(\gamma, v_1) + \phi(\gamma, v_2), \tag{10}$$

$$\phi(\gamma, \theta v) = \theta\, \phi(\gamma, v), \tag{11}$$

$$\forall\, v \in V,\ \ \phi(0, v) = 0, \tag{12}$$

$\exists\, k_2, k_3 > 0$ such that $\forall\,\gamma,\,\delta \in L^2_-(\Xi),\,\forall\, v \in V,$

$$|\phi(\gamma, v) - \phi(\delta, v)| \leq k_2 \|\gamma - \delta\|_{L^2(\Xi)} \, \|v\|_U, \tag{13}$$

$$|\phi(\gamma, v) - \phi(\delta, v)| \leq k_3 \|\gamma - \delta\|_{X'} \, \|v\|, \tag{14}$$

$$\exists\, k_4 > 0 \text{ such that } \|\gamma_1 - \gamma_2\|_{X'} \leq k_4(\|u_1 - u_2\| + \|f_1 - f_2\|), \tag{15}$$

for all $\gamma_{1,2} \in L^2_-(\Xi),\ u_{1,2}, f_{1,2}, d_{1,2} \in V$ verifying

$$(Q_1)\ \ \langle F'(u_1), v - u_1\rangle - \langle \gamma_1, l_0(v - u_1)\rangle_{L^2(\Xi)}$$

$$+\phi(\gamma_1, v - d_1) - \phi(\gamma_1, u_1 - d_1) \geq \langle f_1, v - u_1\rangle \quad \forall\, v \in V,$$

$$(Q_2)\ \ \langle F'(u_2), v - u_2\rangle - \langle \gamma_2, l_0(v - u_2)\rangle_{L^2(\Xi)}$$

$$+\phi(\gamma_2, v - d_2) - \phi(\gamma_2, u_2 - d_2) \geq \langle f_2, v - u_2\rangle \quad \forall\, v \in V,$$

and we assume that $k_3 k_4 < \alpha$. (16)

Let $f \in W^{1,2}(0,T;V)$, $u^0 \in V$, $\lambda^0 \in \Lambda(I(u^0))$ be given and satisfy the following compatibility condition:

$$\langle F'(u^0), v - u^0\rangle - \langle \lambda^0, l_0(v-u^0)\rangle_{L^2(\Xi)} \qquad (17)$$

$$+\phi(\lambda^0, v) - \phi(\lambda^0, u^0) \geq \langle f(0), v - u^0\rangle \quad \forall\, v \in V.$$

Consider the following problem.
Problem *Q*: Find $u \in W^{1,2}(0,T;V)$, $\lambda \in W^{1,2}(0,T;X')$ such that $u(0) = u^0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda(I(u(t)))$ for almost all $t \in (0,T)$, and

$$\langle F'(u), v - \dot{u}\rangle - \langle \lambda, l_0(v-\dot{u})\rangle_{L^2(\Xi)} + \phi(\lambda, v) \qquad (18)$$

$$-\phi(\lambda, \dot{u}) \geq \langle f, v - \dot{u}\rangle \quad \forall\, v \in V \quad \text{a.e. on } (0,T).$$

Incremental formulations

For $n \in N^*$, we set $\Delta t := T/n,\ t_i := i\,\Delta t,\ i = 0, 1, ..., n$. If θ is a continuous function of $t \in [0, T]$ valued in some vector space, we use the notations $\theta^i := \theta(t_i)$ unless $\theta = u$, and if $\varpi^i,\ \forall i \in \{0, 1, ..., n\}$, are elements of some vector space, then we set

$$\partial\varpi^i := \frac{\varpi^{i+1} - \varpi^i}{\Delta t},\ \ \Delta\varpi^i := \varpi^{i+1} - \varpi^i \quad \forall i \in \{0, 1, ..., n-1\}.$$

We approximate the problem Q using the following sequence of incremental problems $(Q^{i,n})_{i=0,1,\dots,n-1}$.

Problem $Q^{i,n}$: Find $u^{i+1} \in V,\ \lambda^{i+1} \in \Lambda(l(u^{i+1}))$ such that

$$\langle F'(u^{i+1}), v - \partial u^i\rangle - \langle\lambda^{i+1}, l_0(v - \partial u^i)\rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v) \tag{19}$$

$$-\phi(\lambda^{i+1}, \partial u^i) \geq \langle f^{i+1}, v - \partial u^i\rangle \quad \forall v \in V.$$

For all $i \in \{0, 1, ..., n-1\}$ the problem $Q^{i,n}$ is equivalent to the following implicit variational inequality:

Problem $\hat{Q}^{i,n}$: Find $u^{i+1} \in V,\ \lambda^{i+1} \in \Lambda(l(u^{i+1}))$ such that

$$\langle F'(u^{i+1}), v - u^{i+1}\rangle - \langle\lambda^{i+1}, l_0(v - u^{i+1})\rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v - u^i) \tag{20}$$

$$-\phi(\lambda^{i+1}, u^{i+1} - u^i) \geq \langle f^{i+1}, v - u^{i+1}\rangle \quad \forall v \in V.$$

Let us define the following functions:

$$u_n(0) = \hat{u}_n(0) = u^0,\ \lambda_n(0) = \lambda^0,\ f_n(0) = f^0 \text{ and}$$
$$\forall\, i \in \{0, 1, ..., n-1\},\ \forall\, t \in (t_i, t_{i+1}],$$
$$u_n(t) = u^{i+1},\ \lambda_n(t) = \lambda^{i+1},$$
$$\hat{u}_n(t) = u^i + (t - t_i)\partial u^i,$$
$$\hat{\lambda}_n(t) = \lambda^i + (t - t_i)\partial\lambda^i,\ f_n(t) = f^{i+1}.$$

Then for all $n \in N^*$ each of the sequences of inequalities $(Q^{i,n})_{i=0,1,\ldots,n-1}$, $(\hat{Q}^{i,n})_{i=0,1,\ldots,n-1}$ is equivalent to the following incremental formulation.
Problem Q^n : Find $u_n \in L^2(0,T;V)$, $\lambda_n \in L^2(\Xi_T)$ such that $\lambda_n(t) \in \Lambda(l(u_n(t)))$ $\forall\, t \in (0,T)$ and

$$\begin{aligned} &\langle F'(u_n(t)), v - \frac{d}{dt}\hat{u}_n(t)\rangle - \langle \lambda_n(t), l_0(v - \frac{d}{dt}\hat{u}_n(t))\rangle_{L^2(\Xi)} \\ &\qquad + \phi(\lambda_n(t), v) - \phi(\lambda_n(t), \frac{d}{dt}\hat{u}_n(t)) \qquad (21) \\ &\geq \langle f_n(t), v - \frac{d}{dt}\hat{u}_n(t)\rangle \quad \forall\, v \in V, \text{ a.e. on } (0,T). \end{aligned}$$

We prove the existence of a fixed point of the multifunction Φ^i by using a corollary of the Ky Fan's fixed point theorem in the particular case of a reflexive Banach space.

Definition

Let Y be a reflexive Banach space, D a weakly closed set in Y, and $\Phi : D \to 2^Y \setminus \{\emptyset\}$ be a multivalued function. Φ is called sequentially weakly upper semicontinuous if $z_p \rightharpoonup z$, $y_p \in \Phi(z_p)$ and $y_p \rightharpoonup y$ imply $y \in \Phi(z)$.

Proposition

Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y, and $\Phi : D \to 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $\Phi(z)$ is convex for every $z \in D$. Then Φ has a fixed point.

Theorem

Assume that (1 - 5), (7 - 14) hold. Then there exists $\lambda \in L^2_-(\Xi)$ such that $\lambda \in \Phi^i(\lambda)$ and (u^{i+1}, λ^{i+1})=(u_λ, λ) is a solution of the problem $\hat{Q}^{i,n}$.

Existence of a solution to the continuous problem

We now establish some useful estimates independent of n for the solutions of the incremental formulations $\hat{Q}^{i,n}$ and Q^n.

Lemma

Under the above hypotheses, for all $n \in N^*$ and all $i \in \{0, 1, ..., n-1\}$ the following estimates hold:

$$\|u^{i+1}\| \leq C_1(\|\lambda^{i+1}\|_{X'} + \|f^{i+1}\| + \|F'(0)\|),$$

$$\|\Delta u^i\| \leq \frac{k_3}{\alpha}\|\Delta\lambda^i\|_{X'} + \frac{1}{\alpha}\|\Delta f^i\|),$$

$$\|\Delta\lambda^i\|_{X'} \leq k_4(\|\Delta u^i\| + \|\Delta f^i\|),$$

$$\|\Delta u^i\| \leq C_2\|\Delta f^i\|,$$

$$\|\Delta\lambda^i\|_{X'} \leq C_3\|\Delta f^i\|,$$

where $C_2 = \dfrac{k_3 k_4 + 1}{\alpha - k_3 k_4}$, $C_3 = \dfrac{(\alpha + 1)k_4}{\alpha - k_3 k_4}$.

Lemma

For all $n \in N^*$

$$\|u_n(t)\| \leq C_1(\|\lambda_n(t)\|_{X'} + \|f_n(t)\| + \|F'(0)\|) \ \ \forall\, t \in [0, T], \tag{27}$$

$$\|u_n(t) - \hat{u}_n(t)\| \leq \frac{T}{n}\left\|\frac{d}{dt}\hat{u}_n(t)\right\| \leq C_2\left\|f_n(t) - f_n(t - \frac{T}{n})\right\| \tag{28}$$

$$\leq C_2 \int_{t-\frac{T}{n}}^{min\{t+\frac{T}{n},T\}} \|\dot{f}(\tau)\|\, d\tau \ \ \forall\, t \in [0, T],$$

$$\|\lambda_n(t) - \hat{\lambda}_n(t)\|_{X'} \leq \frac{T}{n}\left\|\frac{d}{dt}\hat{\lambda}_n(t)\right\|_{X'} \leq C_3\left\|f_n(t) - f_n(t - \frac{T}{n})\right\| \ \ \forall\, t \in [0, T], \tag{29}$$

$$\|u_n - \hat{u}_n\|_{L^2(0,T;V)} = \frac{T}{n\sqrt{3}}\left\|\frac{d}{dt}\hat{u}_n\right\|_{L^2(0,T;V)} \leq C_2\frac{T}{n\sqrt{3}}\|\dot{f}\|_{L^2(0,T;V)}, \tag{30}$$

$$\|\lambda_n - \hat{\lambda}_n\|_{L^2(0,T;X')} = \frac{T}{n\sqrt{3}}\left\|\frac{d}{dt}\hat{\lambda}_n\right\|_{L^2(0,T;X')} \leq C_3\frac{T}{n\sqrt{3}}\|\dot{f}\|_{L^2(0,T;V)}. \tag{31}$$

Lemma

There exist subsequences of $(u_n, \hat{u}_n)_n$ *and* $(\lambda_n, \hat{\lambda}_n)_n$*, denoted by* $(u_{n_p}, \hat{u}_{n_p})_p$ *and* $(\lambda_{n_p}, \hat{\lambda}_{n_p})_p$*, respectively, and two elements* $u \in W^{1,2}(0,T;V)$*,* $\lambda \in W^{1,2}(0,T;X') \cap L^2(\Xi_T)$ *such that*

$$u_{n_p}(t) \rightharpoonup u(t) \;\; in \;\; V \quad \forall\, t \in [0,T], \tag{32}$$

$$\hat{u}_{n_p} \rightharpoonup u \;\; in \;\; W^{1,2}(0,T;V), \tag{33}$$

$$\lambda_{n_p}(t) \rightharpoonup \lambda(t) \;\; in \;\; X' \quad \forall\, t \in [0,T], \tag{34}$$

$$\lambda_{n_p}, \hat{\lambda}_{n_p} \rightharpoonup \lambda \;\; in \;\; L^2(0,T;L^2(\Xi)), \tag{35}$$

$$\hat{\lambda}_{n_p} \rightharpoonup \lambda \;\; in \;\; W^{1,2}(0,T;X'), \tag{36}$$

$$\liminf_{p\to\infty} \int_0^T \phi(\lambda_{n_p}(t), \frac{d}{dt}\hat{u}_{n_p}(t))\, dt \geq \int_0^T \phi(\lambda(t), \frac{d}{dt}\hat{u}(t))\, dt. \tag{37}$$

Theorem

Under the assumptions (1 - 5), (7 - 16), every convergent subsequence of the previous lemma, $(u_{n_p}, \hat{u}_{n_p})_p$, $(\lambda_{n_p}, \hat{\lambda}_{n_p})_p$, *and their limits* $u \in W^{1,2}(0,T;V)$, $\lambda \in W^{1,2}(0,T;X') \cap L^2(\Xi_T)$ *have the following strong convergence properties*

$$u_{n_p}(t) \to u(t) \ \ in \ \ V \quad \forall\, t \in [0,T], \tag{38}$$

$$\lambda_{n_p}(t) \to \lambda(t) \ \ in \ \ X' \quad \forall\, t \in [0,T], \tag{39}$$

and (u, λ) *is a solution to the problem Q.*

Proof. The sequence $(\hat{Q}^{i,n})_{i=0,1,\dots,n-1}$ implies that for every $t \in [0,T]$

$$\langle F'(u_n(t)), v - u_n(t)\rangle - \langle \lambda_n(t), l_0(v - u_n(t))\rangle_{L^2(\Xi)} \tag{40}$$

$$+\phi(\lambda_n(t), v - u_n(t)) \geq \langle f_n(t), v - u_n(t)\rangle \quad \forall\, v \in V,$$

and taking $v = u$, by (6) we derive

$$F(u(t)) - F(u_{n_p}(t)) - \langle \lambda_{n_p}(t), l_0(u(t) - u_{n_p}(t))\rangle_{L^2(\Xi)} \tag{41}$$

$$+\phi(\lambda_{n_p}(t), u(t) - u_{n_p}(t)) \geq \langle f_{n_p}(t), u(t) - u_{n_p}(t)\rangle + \frac{\alpha}{2}\|u(t) - u_{n_p}(t)\|^2 \ \ \forall\, p \in \mathbb{N}.$$

Using that F is sequentially weakly lower semicontinuous, (7), (13), the compact embeddings $X \subset L^2(\Xi)$, $V \subset U$ and that for all $t \in [0, T]$ $(\lambda_{n_p}(t))_p$ is bounded in $L^2(\Xi)$ by R_0, the previous relation implies

$$\begin{aligned}
&\limsup_{p\to\infty} \frac{\alpha}{2}\|u(t) - u_{n_p}(t)\|^2 \\
&\quad \le F(u(t)) + \limsup_{p\to\infty}(-F(u_{n_p}(t))) + \lim_{p\to\infty} |\langle \lambda_{n_p}(t), l_0(u(t) - u_{n_p}(t))\rangle_{L^2(\Xi)}| \\
&\quad + \lim_{p\to\infty} \phi(\lambda_{n_p}(t), u(t) - u_{n_p}(t)) - \lim_{p\to\infty} \langle f_{n_p}(t), u(t) - u_{n_p}(t)\rangle \\
&\quad \le F(u(t)) - \liminf_{p\to\infty} F(u_{n_p}(t)) + \lim_{p\to\infty} \|\lambda_{n_p}(t)\|_{L^2(\Xi)} \|l_0(u(t) - u_{n_p}(t))\|_{L^2(\Xi)} \\
&\quad + \lim_{p\to\infty} k_2\|\lambda_{n_p}(t)\|_{L^2(\Xi)} \|u(t) - u_{n_p}(t)\|_U - \lim_{p\to\infty} \langle f_{n_p}(t), u(t) - u_{n_p}(t)\rangle \\
&\quad = F(u(t)) - \liminf_{p\to\infty} F(u_{n_p}(t)) \le 0,
\end{aligned}$$

which proves (38).

Now, we shall use the following compactness theorem proved by Simon (1987).

Theorem

Let $\hat{X}$, $\hat{U}$ and $\hat{Y}$ be three Banach spaces such that $\hat{X} \subset \hat{U} \subset \hat{Y}$ with compact embedding from $\hat{X}$ into $\hat{U}$.
(i) Let $\mathcal{G}$ be bounded in $L^p(0, T; \hat{X})$, where $1 \leq p < \infty$, and $\partial\mathcal{G}/\partial t := \{\dot{f};\ f \in \mathcal{G}\}$ be bounded in $L^1(0, T; \hat{Y})$. Then $\mathcal{G}$ is relatively compact in $L^p(0, T; \hat{U})$.
(ii) Let $\mathcal{G}$ be bounded in $L^\infty(0, T; \hat{X})$ and $\partial\mathcal{G}/\partial t$ be bounded in $L^r(0, T; \hat{Y})$, where $r > 1$. Then $\mathcal{G}$ is relatively compact in $C([0, T]; \hat{U})$.

By the Simon's theorem with $\mathcal{G} = (\hat{\lambda}_{n_p})_p$, $\hat{X} = L^2(\Xi)$, $\hat{U} = H^{\iota - 1/2}(\Xi)$, $\hat{Y} = X'$, $r = 2$, $0 < \iota < \frac{1}{2}$, it follows that

$$\hat{\lambda}_{n_p} \to \lambda \ \text{ in } \ C([0, T]; X'), \tag{42}$$

so that by (29) we obtain (39).
It remains to prove that (u, λ) is a solution of the problem Q.

First, since $\lambda_{n_p}(t) \in \Lambda(l(u_{n_p}(t)))$ for all $t \in (0, T)$, we have

$$\int_\omega \underline{\kappa} \circ l(u_{n_p}) \leq \int_\omega \lambda_{n_p} \leq \int_\omega \overline{\kappa} \circ l(u_{n_p}), \tag{43}$$

for every measurable subset $\omega \subset \Xi_T$ and for all $p \in \mathbb{N}$.
Using (38), (8), the semi-continuity of $\underline{\kappa}$ and $\overline{\kappa}$, the relations (1), (2), (35), which implies the convergence property $\int_\omega \lambda_{n_p} \to \int_\omega \lambda$, and passing to limits according to Fatou's lemma, we obtain

$$\int_\omega \underline{\kappa} \circ l(u) \leq \int_\omega \lambda \leq \int_\omega \overline{\kappa} \circ l(u), \tag{44}$$

for every measurable subset $\omega \subset \Xi_T$, which implies $\lambda(t) \in \Lambda(l(u(t)))$ for almost all $t \in (0, T)$.
Second, integrating both sides in (21) over $[0, T]$ and passing to the limit, by the relations (38), (39), (33), (37), it follows that for all $v \in L^2(0, T; V)$

$$\int_0^T \langle F'(u(t)), v(t) - \dot{u}(t) \rangle dt - \int_0^T \langle \lambda(t), l_0(v(t) - \dot{u}(t)) \rangle_{L^2(\Xi)} dt$$
$$+ \int_0^T \phi(\lambda(t), v(t)) dt - \int_0^T \phi(\lambda(t), \dot{u}(t)) dt \geq \int_0^T \langle f(t), v(t) - \dot{u}(t) \rangle dt.$$

By Lebesgue's theorem, it follows that (u, λ) is a solution of the problem Q.

Applications to two quasistatic contact problems

Consider an elastic body occupying the set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where Γ_1, Γ_2, Γ_3 are open, disjoint parts of Γ and $\text{meas}(\Gamma_1) > 0$. Assume the small deformation hypothesis and that the inertial effects are negligible.
Denote by $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ the displacement field, by $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor and by $\boldsymbol{\sigma}$ the stress tensor, with the components $u = (u_i)$, $\varepsilon = (\varepsilon_{ij})$ and $\sigma = (\sigma_{ij})$, respectively. We use the classical decompositions $\boldsymbol{u} = u_N\boldsymbol{n} + \boldsymbol{u}_T$, $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$, $\boldsymbol{\sigma n} = \sigma_N \boldsymbol{n} + \boldsymbol{\sigma}_T$, $\sigma_N = (\boldsymbol{\sigma n}) \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the outward normal unit vector to Γ with the components $n = (n_i)$. The usual summation convention will be used for $i, j, k, l = 1, \ldots, d$.
Consider the Hilbert space $\boldsymbol{V}$ and the closed convex sets $L^2_-(\Gamma_3)$, $\Lambda_1(\zeta)$ as follows:

$$\boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d);\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$

$$L^2_-(\Gamma_3) := \{\delta \in L^2(\Gamma_3);\ \delta \leq 0 \text{ a.e. in } \Gamma_3\},$$

$$\Lambda_1(\zeta) = \{\eta \in L^2_-(\Gamma_3); \underline{\kappa} \circ \zeta \leq \eta \leq \overline{\kappa} \circ \zeta \text{ a.e. in } \Gamma_3\} \ \ \forall \zeta \in L^2(\Gamma_3).$$

Assume that in Ω a body force $\boldsymbol{\varphi}_1 \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^d))$ is prescribed, on Γ_1 the displacement vector equals zero and on Γ_2 a traction $\boldsymbol{\varphi}_2 \in W^{1,2}(0, T; L^2(\Gamma_2; \mathbb{R}^d))$ is applied.

On Γ_3, the contact between the body and a support is possible with the initial gap denoted by g_0 and the gap corresponding to the solution $\boldsymbol{u}$ denoted by $[u_N] := u_N - g_0$. We assume that there exists $\boldsymbol{g} \in \boldsymbol{V}$ such that $g_N = g_0$ on Γ_3.
On the potential contact surface Γ_3, the displacements and the stress vector will satisfy some contact conditions having the following form:

$$\underline{\kappa}([u_N]) \leq \sigma_N \leq \overline{\kappa}([u_N]).$$

Assume that, for all $\gamma, \delta \in L^2_-(\Gamma_3)$ and all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ such that $\gamma \in \Lambda_1([v_N])$, $\delta \in \Lambda_1([w_N])$,

$$\langle \gamma - \delta, v_N - w_N \rangle_{L^2(\Gamma_3)} \leq 0. \tag{45}$$

Example 1. (Friction conditions with controlled normal stress)
Let $M \geq 0$ be a constant and define

$$\underline{\kappa}(s) = \underline{\kappa}_M(s) = \begin{cases} 0 & \text{if} \quad s < 0, \\ -M & \text{if} \quad s \geq 0, \end{cases} \qquad \overline{\kappa}(s) = \overline{\kappa}_M(s) = \begin{cases} 0 & \text{if} \quad s \leq 0, \\ -M & \text{if} \quad s > 0. \end{cases}$$

The classical Signorini's conditions correspond formally to $M = +\infty$.
Example 2. (Normal compliance conditions)
Various normal compliance conditions and friction laws can be obtained if one considers $\underline{\kappa} = \overline{\kappa} = \kappa$, where $\kappa : \mathbb{R} \to \mathbb{R}$ is some negative, decreasing, and bounded Lipschitz continuous function, so that σ_N is given by the relation $\sigma_N = \kappa([u_N])$.
It is easily seen that these two examples verify the condition (45).

Let $\mathcal{F} \geq 0$ be the coefficient of friction, assumed to be a Lipschitz continuous function on Γ, which ensures to belong to the set of the multipliers on $H^{1/2}(\Gamma)$ denoted by $\mathcal{M}$. Therefore the mapping $H^{1/2}(\Gamma) \ni v \mapsto \mathcal{F}v \in H^{1/2}(\Gamma)$ is bounded with norm $\|\mathcal{F}\|_{\mathcal{M}}$. In order to describe the frictional contact conditions on Γ_3, we define

$$\forall\, \boldsymbol{l} \in \boldsymbol{V},\ \ \boldsymbol{S_l} := \{\boldsymbol{v} \in \boldsymbol{V};\ \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi})dx = \langle \boldsymbol{l}, \boldsymbol{\psi}\rangle_{\boldsymbol{V}} \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{V} \text{ such that } \boldsymbol{\psi} = \boldsymbol{0} \text{ a.e. on } \Gamma_3\},$$

$$\boldsymbol{L} \in \boldsymbol{V},\ \ \langle \boldsymbol{L}, \boldsymbol{w}\rangle_{\boldsymbol{V}} = \langle \boldsymbol{\varphi}_1, \boldsymbol{w}\rangle_{L^2(\Omega;\,\mathbb{R}^d)} + \langle \boldsymbol{\varphi}_2, \boldsymbol{w}\rangle_{L^2(\Gamma_2;\,\mathbb{R}^d)} \quad \forall\, \boldsymbol{w} \in \boldsymbol{V},$$

$$\forall\, \boldsymbol{v} \in \boldsymbol{S_L},\ \ \langle \sigma_N(\boldsymbol{v}), w\rangle_\Gamma = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\bar{\boldsymbol{w}})dx - \langle \boldsymbol{L}, \bar{\boldsymbol{w}}\rangle_{\boldsymbol{V}} \ \ \forall\, w \in H^{1/2}(\Gamma),$$

where $\langle \cdot\,, \cdot\rangle_\Gamma$ denotes the duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, $\bar{\boldsymbol{w}} \in \boldsymbol{V}$ satisfies $\bar{\boldsymbol{w}}_T = \boldsymbol{0}$ a.e. on Γ_3, $\bar{\boldsymbol{w}}_N = w$ a.e. on Γ_3. It is easy to verify that for all $\boldsymbol{v} \in \boldsymbol{S_L}$ $\sigma_N(\boldsymbol{v})$ depends only on the values of w on Γ_3 and not on the choices of $\bar{\boldsymbol{w}}$ having the above properties.

A contact problem for a nonlinear Hencky material

Assume that the elastic body satisfies the following nonlinear Hencky-Mises constitutive equation:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \hat{\boldsymbol{\sigma}}(\boldsymbol{u}) = (k - \frac{2}{3}\,\mu(\hat{\gamma}(\boldsymbol{u})))(\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\boldsymbol{I} + 2\,\mu(\hat{\gamma}(\boldsymbol{u}))\,\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where k is the constant bulk modulus, μ is a continuously differentiable function in $[\,0, +\infty)$ satisfying

$$0 < \mu_0 \leq \mu(r) \leq \frac{3}{2}\,k,\; 0 < \mu_1 \leq \mu(r) + 2\,\frac{\partial\mu(r)}{\partial r}\,r \leq \mu_2,\; \forall\, r \geq 0, \tag{46}$$

and, for all $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{V}$,

$$\hat{\gamma}(\boldsymbol{u}) := \hat{\gamma}(\boldsymbol{u}, \boldsymbol{u}),\; \hat{\gamma}(\boldsymbol{u}, \boldsymbol{v}) = -\frac{2}{3}\,\vartheta(\boldsymbol{u})\,\vartheta(\boldsymbol{v}) + 2\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}),\; \vartheta(\boldsymbol{u}) := \mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathrm{div}\,\boldsymbol{u}.$$

Consider the following quasistatic contact problem with Coulomb friction.
Problem P_1^c : Find $\boldsymbol{u}$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and, for all $t \in (0, T)$,

$$\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = -\varphi_1 \text{ in } \Omega, \tag{47}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \hat{\boldsymbol{\sigma}}(\boldsymbol{u}) \text{ in } \Omega, \tag{48}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_1, \ \boldsymbol{\sigma n} = \varphi_2 \text{ on } \Gamma_2, \tag{49}$$

$$\underline{\kappa}([u_N]) \le \sigma_N \le \overline{\kappa}([u_N]) \text{ on } \Gamma_3, \tag{50}$$

$$|\boldsymbol{\sigma}_T| \le \mathcal{F}\,|\sigma_N| \text{ and } \tag{51}$$

$$\dot{\boldsymbol{u}}_T \ne \boldsymbol{0} \Rightarrow \boldsymbol{\sigma}_T = -\mathcal{F}|\sigma_N|\frac{\dot{\boldsymbol{u}}_T}{|\dot{\boldsymbol{u}}_T|} \text{ on } \Gamma_3.$$

Let $F_1 : \boldsymbol{V} \to \mathbb{R}$ be defined by

$$F_1(\boldsymbol{v}) = \frac{1}{2}k\int_\Omega \vartheta^2(\boldsymbol{v})dx + \frac{1}{2}\int_\Omega \left(\int_0^{\hat{\gamma}(\boldsymbol{v})} \mu(r)dr\right)dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{52}$$

and $J : L^2_-(\Gamma_3) \times \boldsymbol{V} \to \mathbb{R}$ be defined by

$$J(\gamma, \boldsymbol{v}) = -\int_{\Gamma_3} \mathcal{F}\,\gamma\,|\boldsymbol{v}_T|ds \quad \forall\, \gamma \in L^2_-(\Gamma_3),\ \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{53}$$

F_1 is differentiable on $\boldsymbol{V}$ and for all $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{V}$

$$\langle F_1'(\boldsymbol{u}), \boldsymbol{v}\rangle_{\boldsymbol{V}} = \int_\Omega [(k - \frac{2}{3}\,\mu(\hat{\gamma}(\boldsymbol{u})))\,\vartheta(\boldsymbol{u})\,\vartheta(\boldsymbol{v}) + 2\,\mu(\hat{\gamma}(\boldsymbol{u}))\,\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})]dx. \tag{54}$$

Let $\boldsymbol{u}_0 \in \boldsymbol{V}$, $\lambda^0 \in \Lambda_1([u_{0N}])$ satisfy the following compatibility condition:

$$\begin{aligned}&\langle F_1'(\boldsymbol{u}_0), \boldsymbol{v} - \boldsymbol{u}_0\rangle_{\boldsymbol{V}} - \langle \lambda^0, v_N - u_{0N}\rangle_{L^2(\Gamma_3)} \\ &\qquad + J(\lambda^0, \boldsymbol{v}) - J(\lambda^0, \boldsymbol{u}_0) \geq \langle \boldsymbol{L}(0), \boldsymbol{v} - \boldsymbol{u}_0\rangle_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.\end{aligned} \tag{55}$$

We have the following variational formulation of problem P_1^c.
Problem $\boldsymbol{P_1^v}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in W^{1,2}(0,T;H^{-1/2}(\Gamma))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda_1([u_N(t)])$ for almost all $t \in (0,T)$, and

$$\begin{aligned}&\langle F_1'(\boldsymbol{u}), \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle_{\boldsymbol{V}} - \langle \lambda, v_N - \dot{u}_N\rangle_{L^2(\Gamma_3)} + J(\lambda, \boldsymbol{v}) \\ &\qquad - J(\lambda, \dot{\boldsymbol{u}}) \geq \langle \boldsymbol{L}, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \quad \text{a.e. on } (0,T).\end{aligned} \tag{56}$$

The Lagrange multiplier $\lambda \in L^2(\Gamma_3)$ satisfies the relation $\sigma_N = \lambda$ in $H^{-1/2}(\Gamma)$ that is

$$\langle \sigma_N(\boldsymbol{u}), w \rangle_\Gamma = \langle \lambda, w \rangle_{L^2(\Gamma_3)} \quad \forall\, w \in H^{1/2}(\Gamma).$$

Taking $\Xi = \Gamma_3$, $\Lambda = \Lambda_1$, $V = \boldsymbol{V}$, $U = H^\iota(\Omega;\mathbb{R}^d)$, $1 > \iota > \frac{1}{2}$, $X = H^{1/2}(\Gamma)$, $F = F_1$, $\phi = J$, $f = \boldsymbol{L}$, and $l_0(\boldsymbol{v}) = v_N$, $l(\boldsymbol{v}) = [v_N] = v_N - g_0$ $\forall \boldsymbol{v} \in \boldsymbol{V}$, it results that the problem P_1^v is a particular case of problem Q.
It is straightforward to verify the assumptions (1 - 5), (7 - 15), and also (16) if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small, so that by the main theorem we obtain the following existence result.

Proposition

Under the previous assumptions and if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small there exists a solution to problem P_1^v.

A contact problem for a linearly elastic body

Let $\boldsymbol{\mathcal{A}}$ denote the elasticity tensor, with the components $\mathcal{A} = (\mathcal{A}_{ijkl})$ satisfying the following classical symmetry and ellipticity conditions: $\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij} \in L^\infty(\Omega)$, $\forall\, i, j, k, l = 1, \ldots, d$, $\exists\, \alpha_{\mathcal{A}} > 0$ such that $\mathcal{A}_{ijkl}\tau_{ij}\tau_{kl} \geq \alpha_{\mathcal{A}}\, \tau_{ij}\tau_{ij}$ $\forall\, \tau = (\tau_{ij})$ satisfying $\tau_{ij} = \tau_{ji}, \forall\, i, j = 1, \ldots, d$.
Consider the following elastic contact problem with Coulomb friction.
Problem $\boldsymbol{P_2^c}$: Find $\boldsymbol{u}$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, satisfying

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) \ \text{ in } \ \Omega, \tag{57}$$

and (47), (49 - 51) for all $t \in (0, T)$.
Let us define the bilinear and symmetric mapping $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\, dx = \int_\Omega \mathcal{A}_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w)dx \quad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}. \tag{58}$$

The form a is continuous on $\boldsymbol{V} \times \boldsymbol{V}$ and, since $\text{meas}(\Gamma_1) > 0$, by Korn's inequality is also $\boldsymbol{V}$-elliptic.

Let $\boldsymbol{u}_0 \in \boldsymbol{V}$, $\lambda^0 \in \Lambda_1([u_{0N}])$ satisfy the following compatibility condition:

$$a(\boldsymbol{u}_0, \boldsymbol{v} - \boldsymbol{u}_0) - \langle \lambda^0, v_N - u_{0N} \rangle_{L^2(\Gamma_3)} \tag{59}$$

$$+J(\lambda^0, \boldsymbol{v}) - J(\lambda^0, \boldsymbol{u}_0) \geq \langle \boldsymbol{L}(0), \boldsymbol{v} - \boldsymbol{u}_0 \rangle_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

We have the following variational formulation of problem P_2^c.
Problem $\boldsymbol{P_2^v}$: Find $\boldsymbol{u} \in W^{1,2}(0, T; \boldsymbol{V})$, $\lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda_1([u_N(t)])$ for almost all $t \in (0, T)$, and

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) - \langle \lambda, v_N - \dot{u}_N \rangle_{L^2(\Gamma_3)} + J(\lambda, \boldsymbol{v}) \tag{60}$$

$$-J(\lambda, \dot{\boldsymbol{u}}) \geq \langle \boldsymbol{L}, \boldsymbol{v} - \dot{\boldsymbol{u}} \rangle_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \quad \text{a.e. on } (0, T).$$

The Lagrange multiplier $\lambda \in L^2(\Gamma_3)$ satisfies again the relation $\sigma_N = \lambda$ in $H^{-1/2}(\Gamma)$. Taking Ξ, Λ, V, U, X, ϕ, f, l_0, l as previously and $F(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v})$ $\forall \boldsymbol{v} \in \boldsymbol{V}$, we see that the problem P_2^v is a particular case of problem Q so that by using again the existence theorem one obtains the following existence result.

Proposition

Under the previous assumptions and if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small there exists a solution to problem P_2^v.

Perspectives

- Nonlinear quasistatic and dynamic unilateral contact problems with friction
- More complex contact interaction laws (in quasistatic or dynamic cases)
- Numerical analysis and solution methods